

The INRIA logo is written in a red-to-orange gradient cursive font on a white rounded rectangular background.

Inria

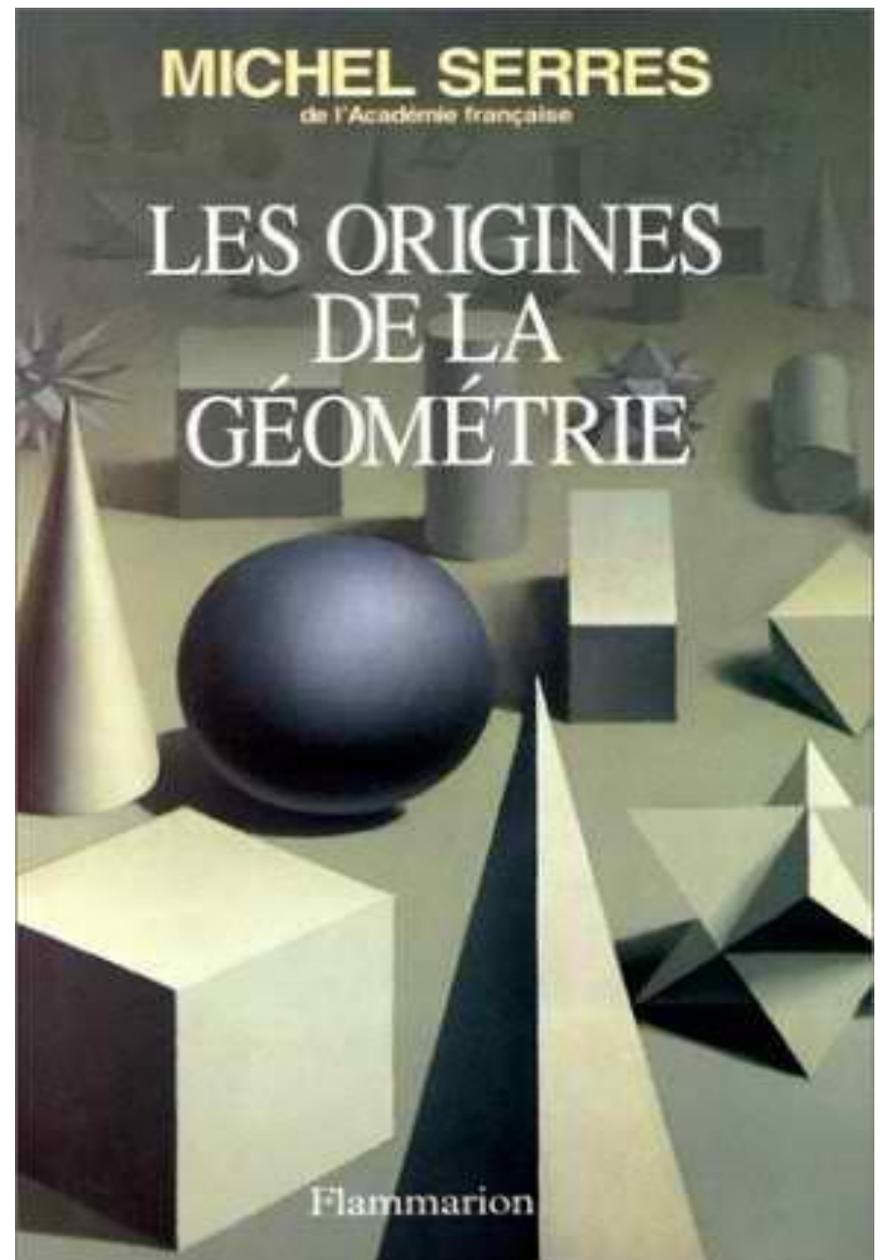
Géométrie numérique

Mesurer des univers réels et virtuels

Bruno Lévy

Bruno Lévy
ALICE Géométrie & Lumière
CENTRE INRIA Nancy Grand-Est

1. Introduction



1. Introduction

GEOMETRIE

1. Introduction

γεωμέτρης

1. Introduction

γεωμέτρης

γῆ

Gê: *la terre*

μέτρον

Metron: *mesurer*

1. Introduction – Mesurer les terres

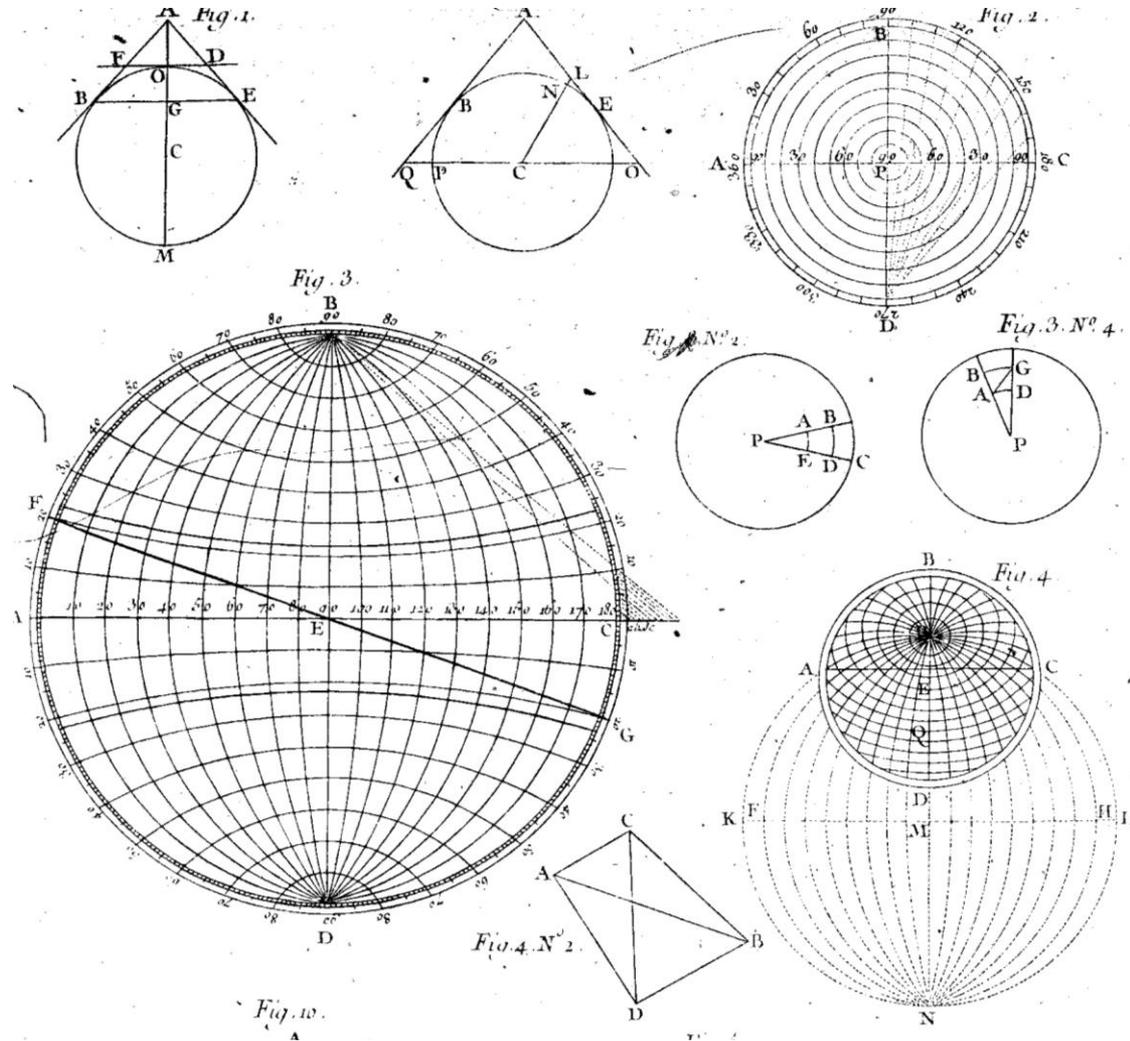


-3000

-1850 “Rhind” papyrus (British Museum)

1. Introduction – Mesurer la terre

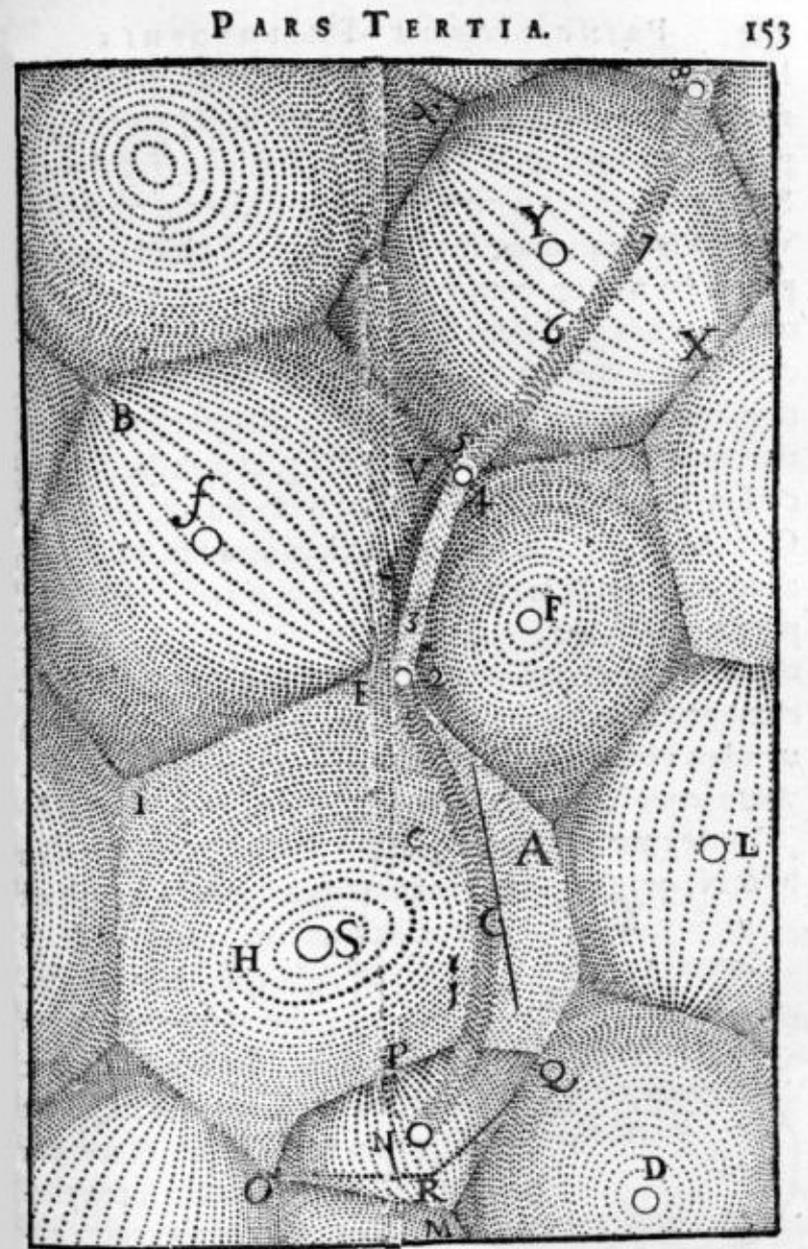
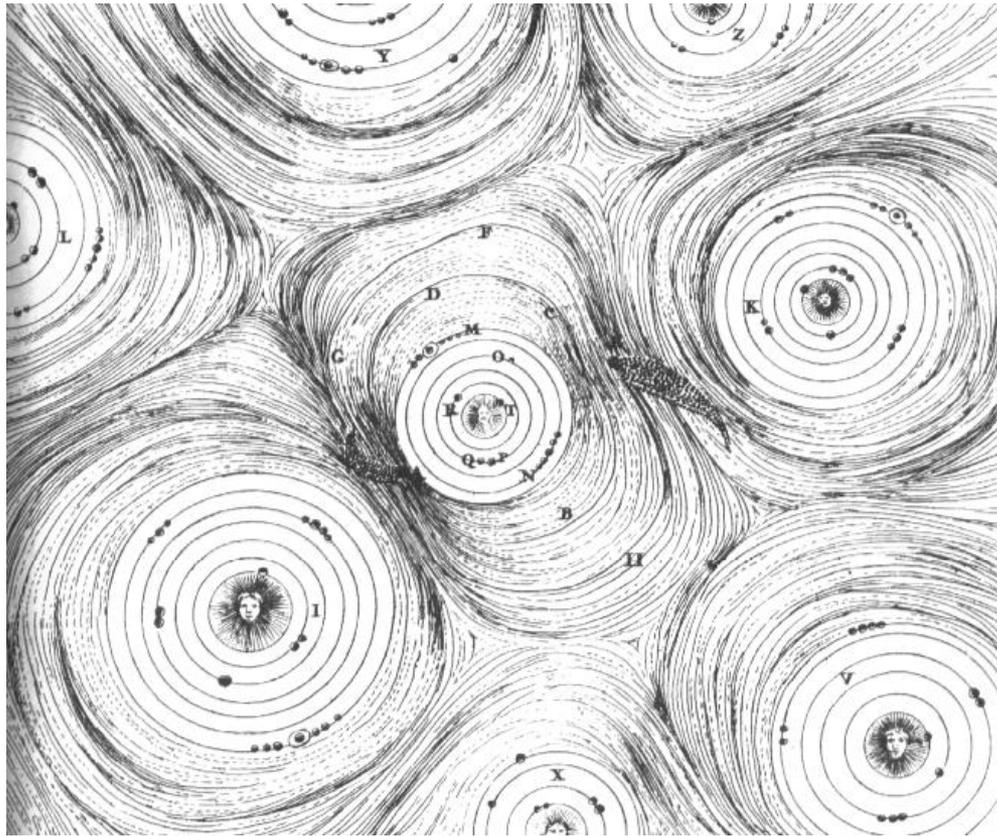
Gérard Mercator
1530



1. Introduction

... Mesurer l'ether*

René Descartes - 1663

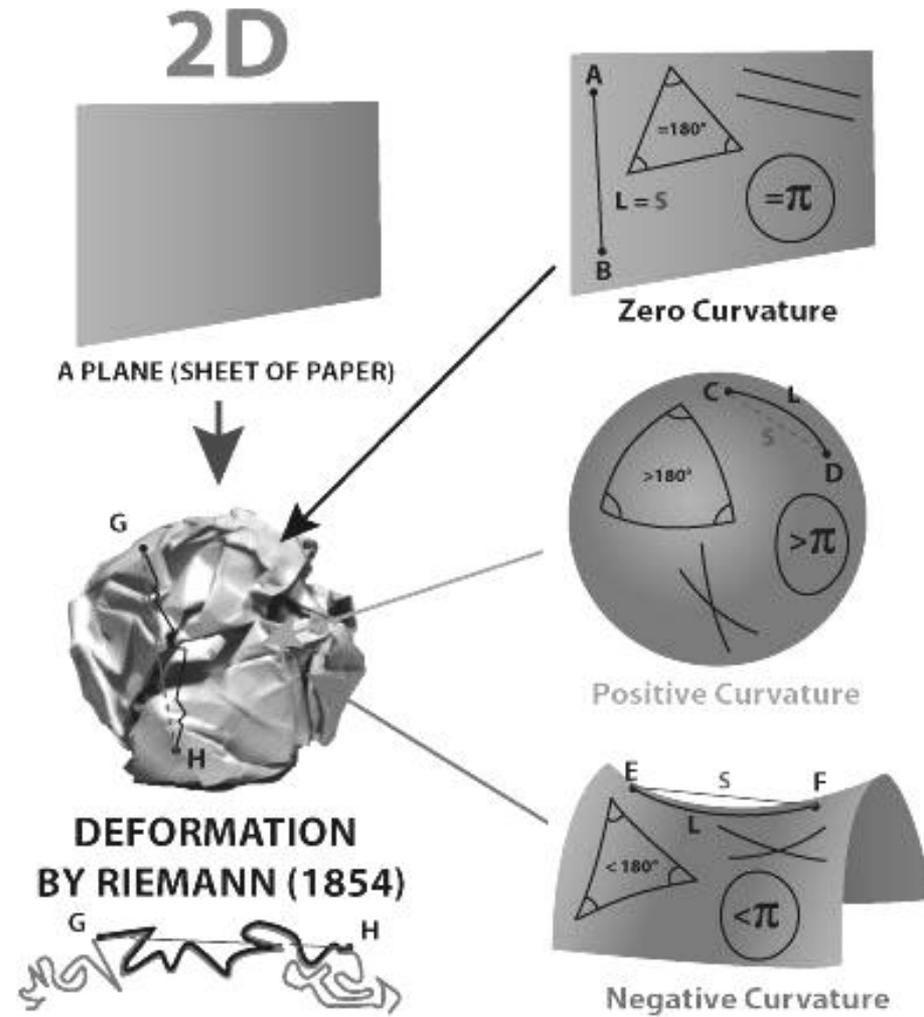
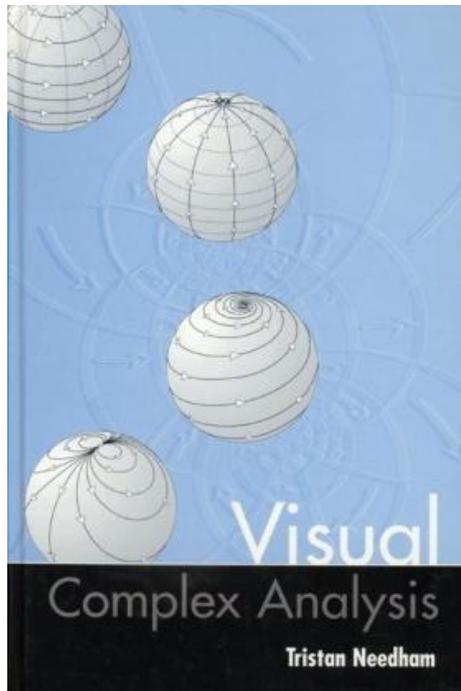


1. Introduction

Géométrie différentielle

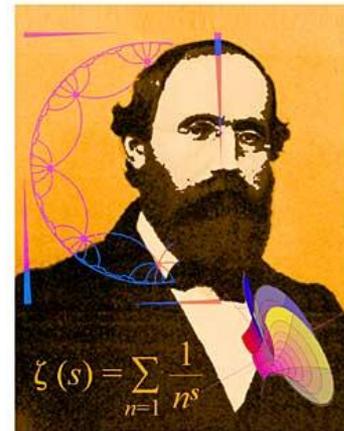
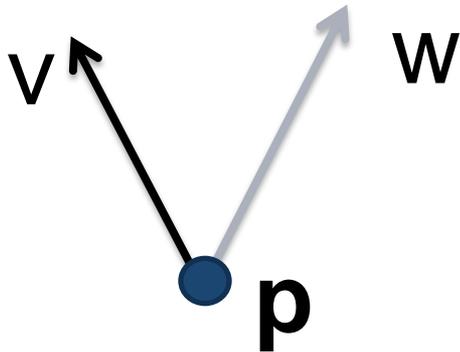
Carl Friedrich Gauss 1800

Bernhard Riemann 1850

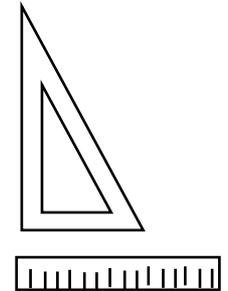


1. Introduction – Géométrie Riemannienne

Le produit scalaire – un outil géométrique



Georg Friedrich Bernhard Riemann. 1826 - 1866

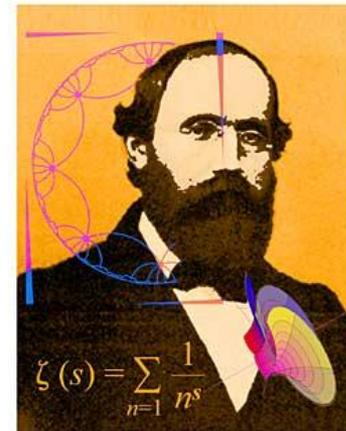
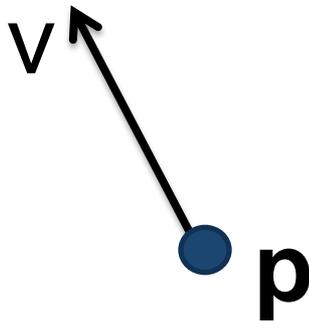


$$v \cdot w = \langle v, w \rangle = v^t w$$

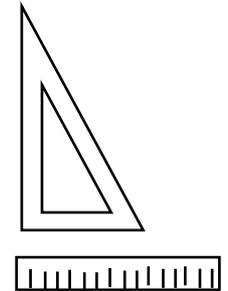
1. Introduction – Géométrie Riemannienne

Le produit scalaire – un outil géométrique

Mesurer des distances



Georg Friedrich Bernhard Riemann. 1826 - 1866

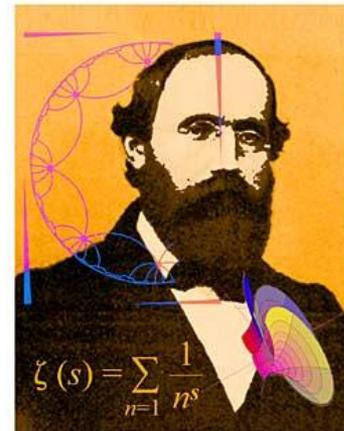
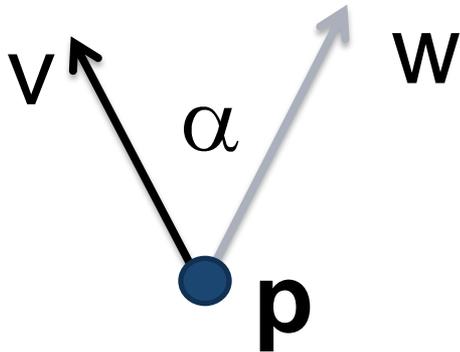


$$\|v\| = \sqrt{\langle v, v \rangle}$$

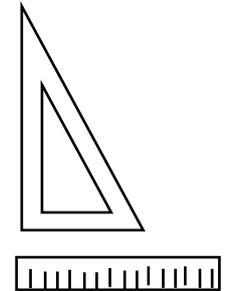
1. Introduction – Géométrie Riemannienne

Le produit scalaire – un outil géométrique

Mesurer des angles



Georg Friedrich Bernhard Riemann. 1826 - 1866



$$\cos(\alpha) = \langle v, w \rangle / \sqrt{\langle v, v \rangle \langle w, w \rangle}$$

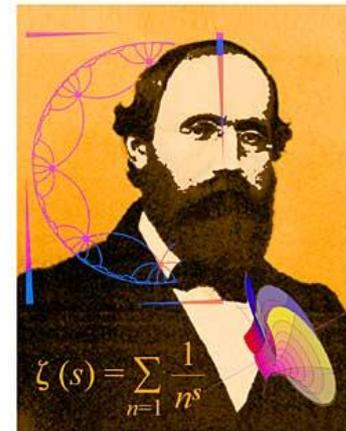
1. Introduction – Géométrie Riemannienne

Le produit scalaire – un outil géométrique

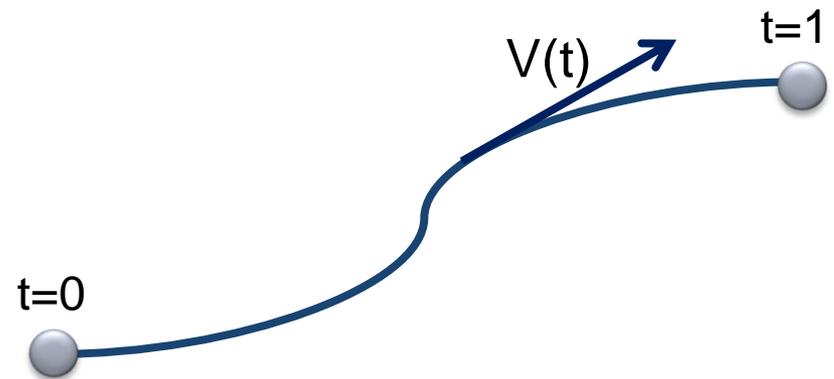
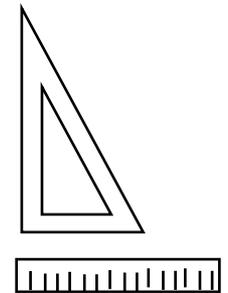
*Mesurer la longueur
d'une courbe*

$$l(C) = \int_{t=0}^1 \|v(t)\| dt$$

$$= \int_{t=0}^1 \sqrt{\langle v(t), v(t) \rangle} dt$$



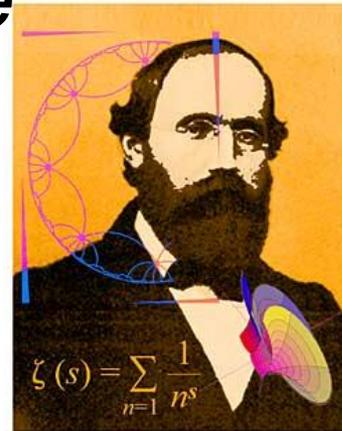
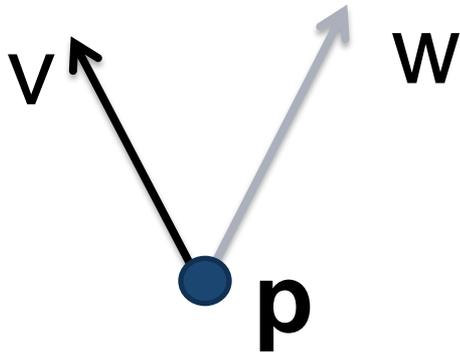
Georg Friedrich Bernhard Riemann. 1826 - 1866



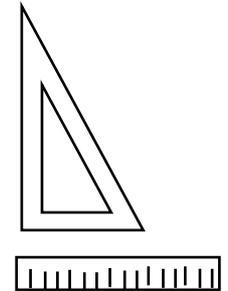
1. Introduction – Géométrie Riemannienne

Le produit scalaire – un outil géométrique

Changer le produit scalaire



Georg Friedrich Bernhard Riemann. 1826 - 1866

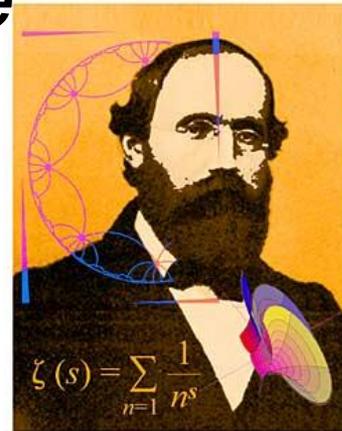
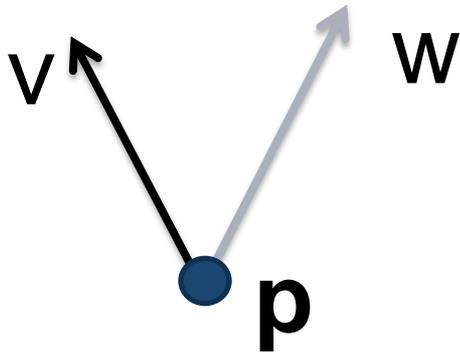


$$v \cdot w = \langle v, w \rangle = v^t w = v^t \text{Id} w$$

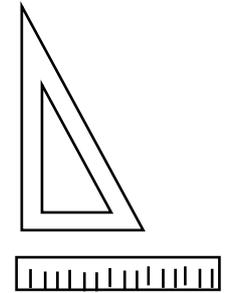
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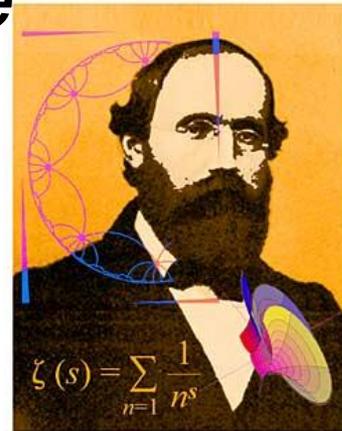
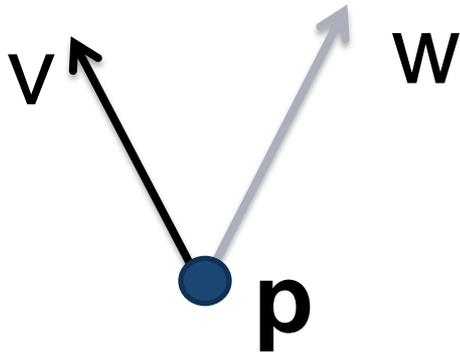
$$v \cdot w = \langle v, w \rangle = v^t \text{Id } w$$

$$\langle v, w \rangle_G = v^t G(p) w$$

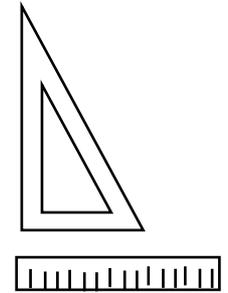
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Le produit scalaire – un outil géométrique

Changer le produit scalaire



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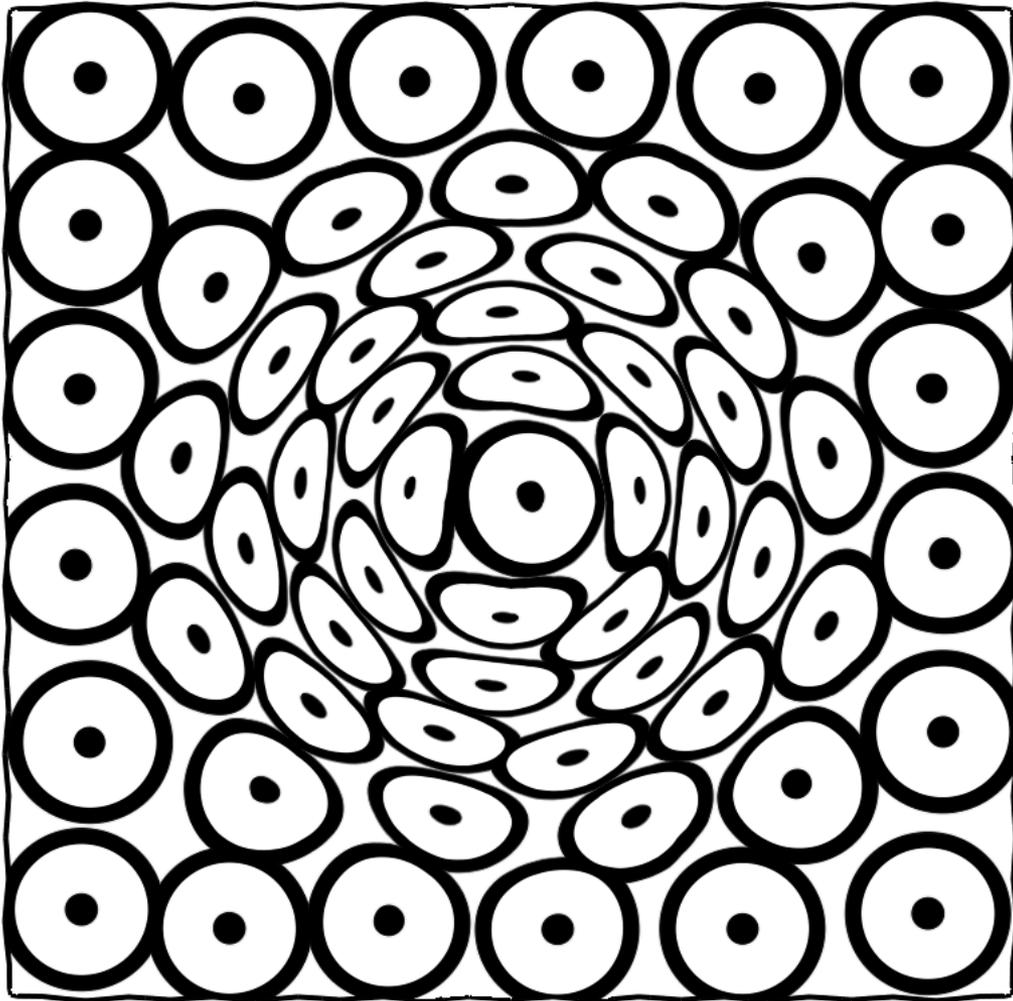


$$v \cdot w = \langle v, w \rangle = v^t \text{Id } w$$

$$\langle v, w \rangle_G = v^t \mathbf{G}(p) w$$

Une matrice 2x2 symétrique qui dépend de p

1. Introduction – Géométrie Riemannienne



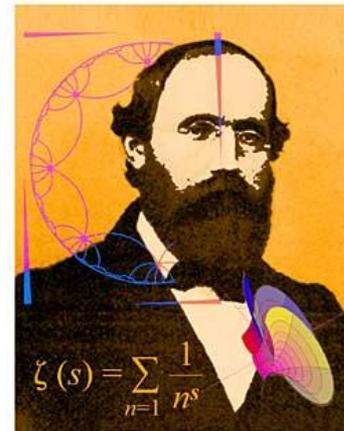
$$G(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

1. Introduction – Géométrie Riemannienne

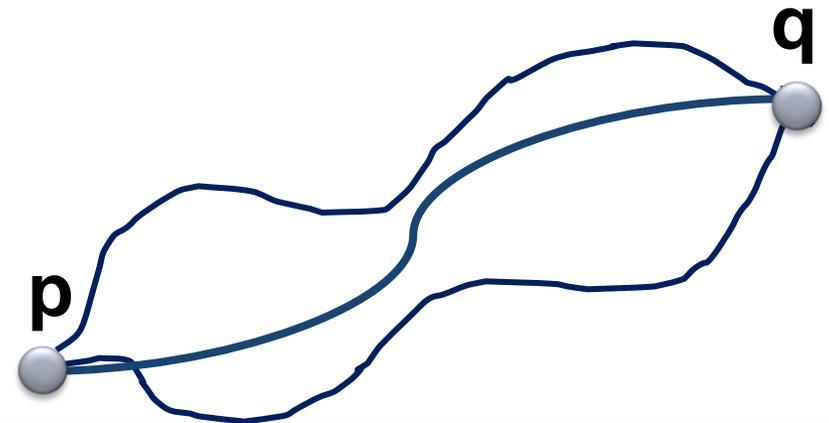
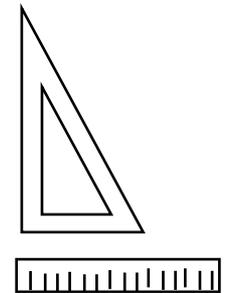
Le produit scalaire – un outil géométrique

*Distance anisotrope
entre deux points p et q*

$d_G(p, q)$ = longueur de la courbe
la plus courte entre p et q

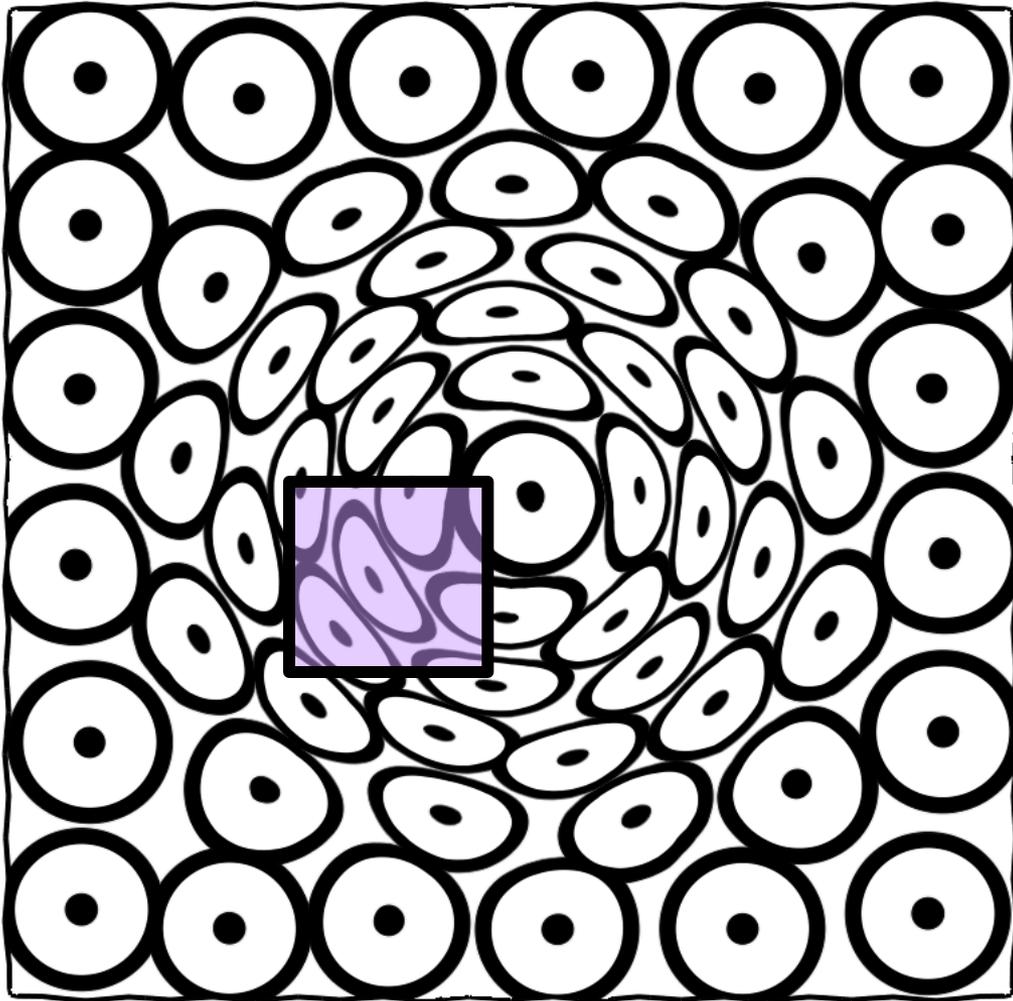


Géométrie Riemannienne. 1826 - 1866



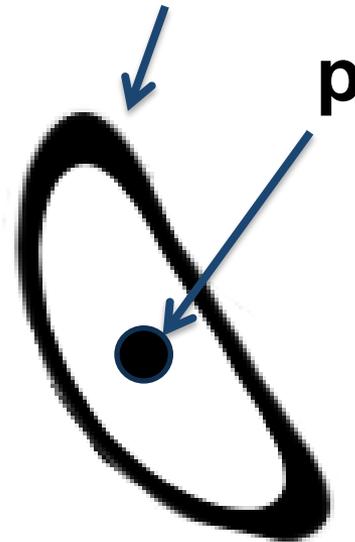
$$l_G(C) = \int_{t=0}^1 \sqrt{v(t)^t G(t) v(t)} dt$$

1. Introduction – Géométrie Riemannienne



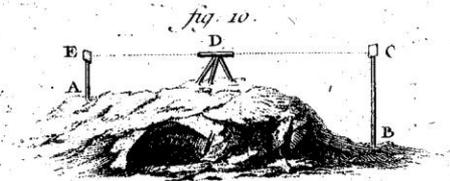
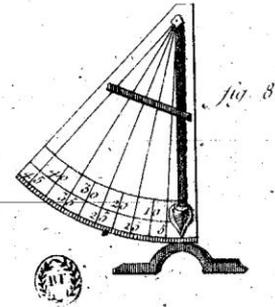
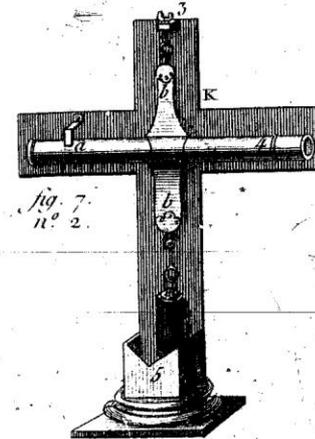
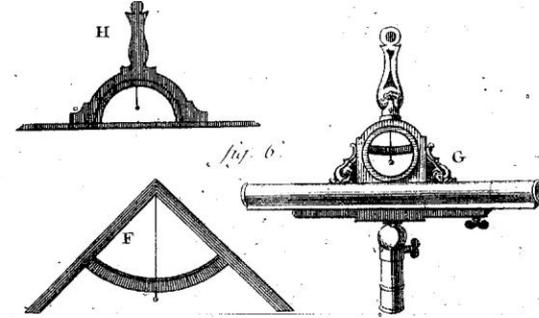
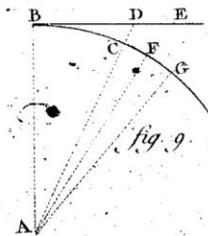
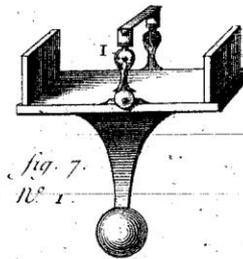
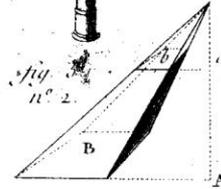
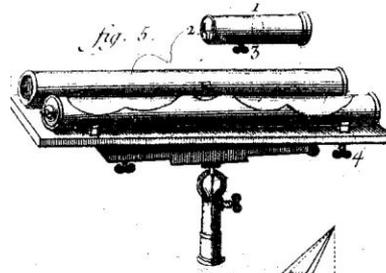
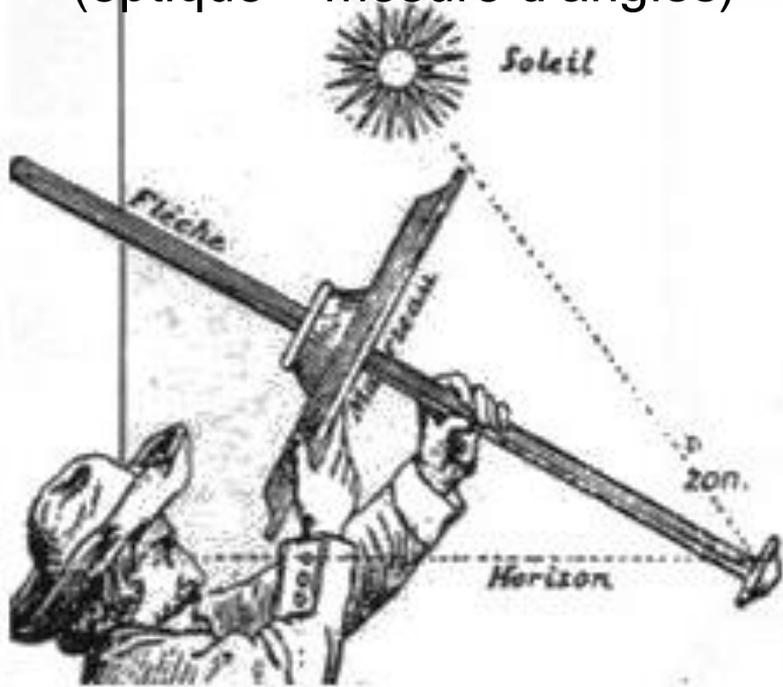
$$G(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

$$\{ q \mid d_G(p, q) = 1 \}$$

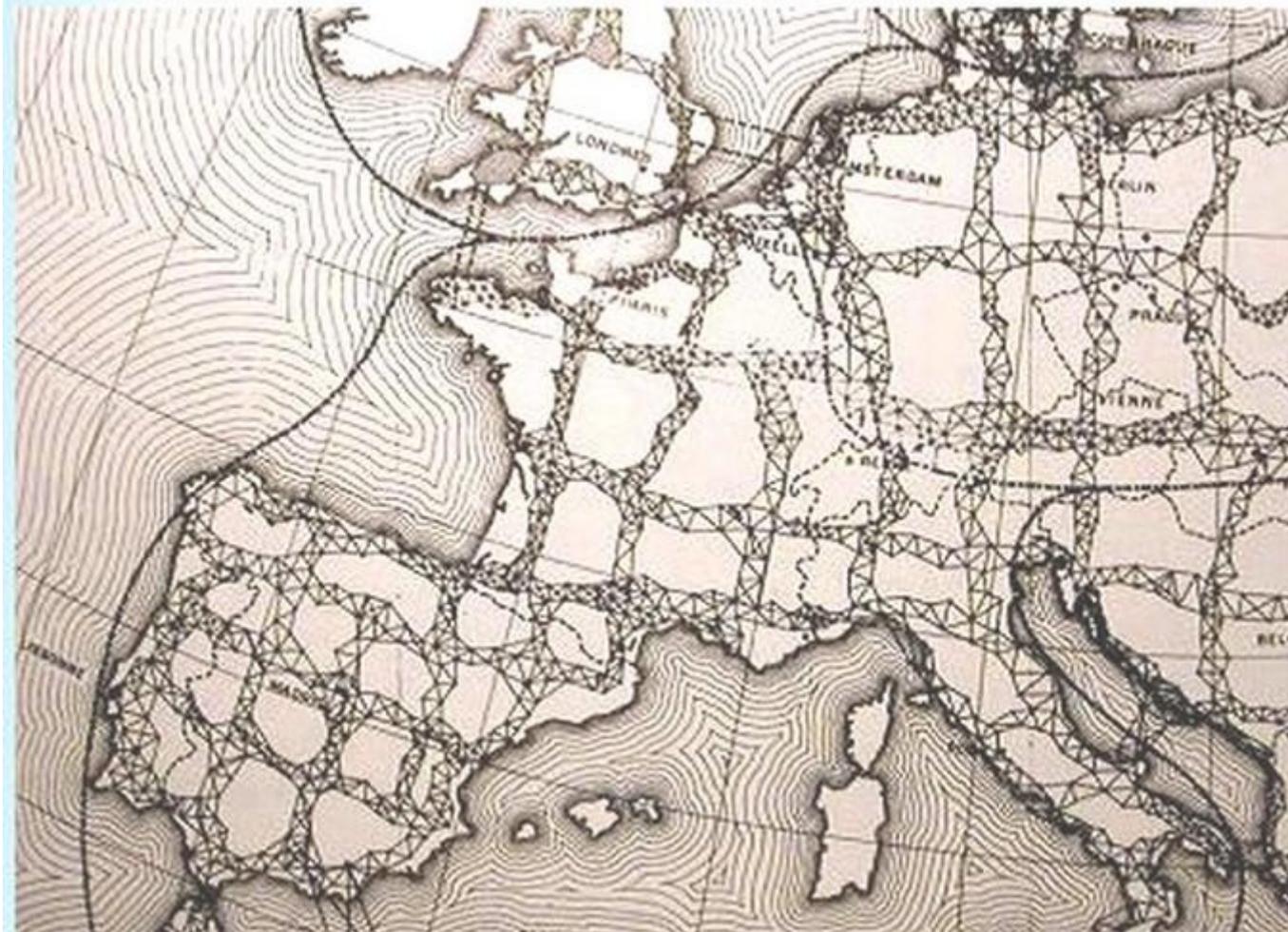


1. Introduction

XVIIIème:
Grands progrès -
instruments
(optique – mesure d'angles)



1. Introduction



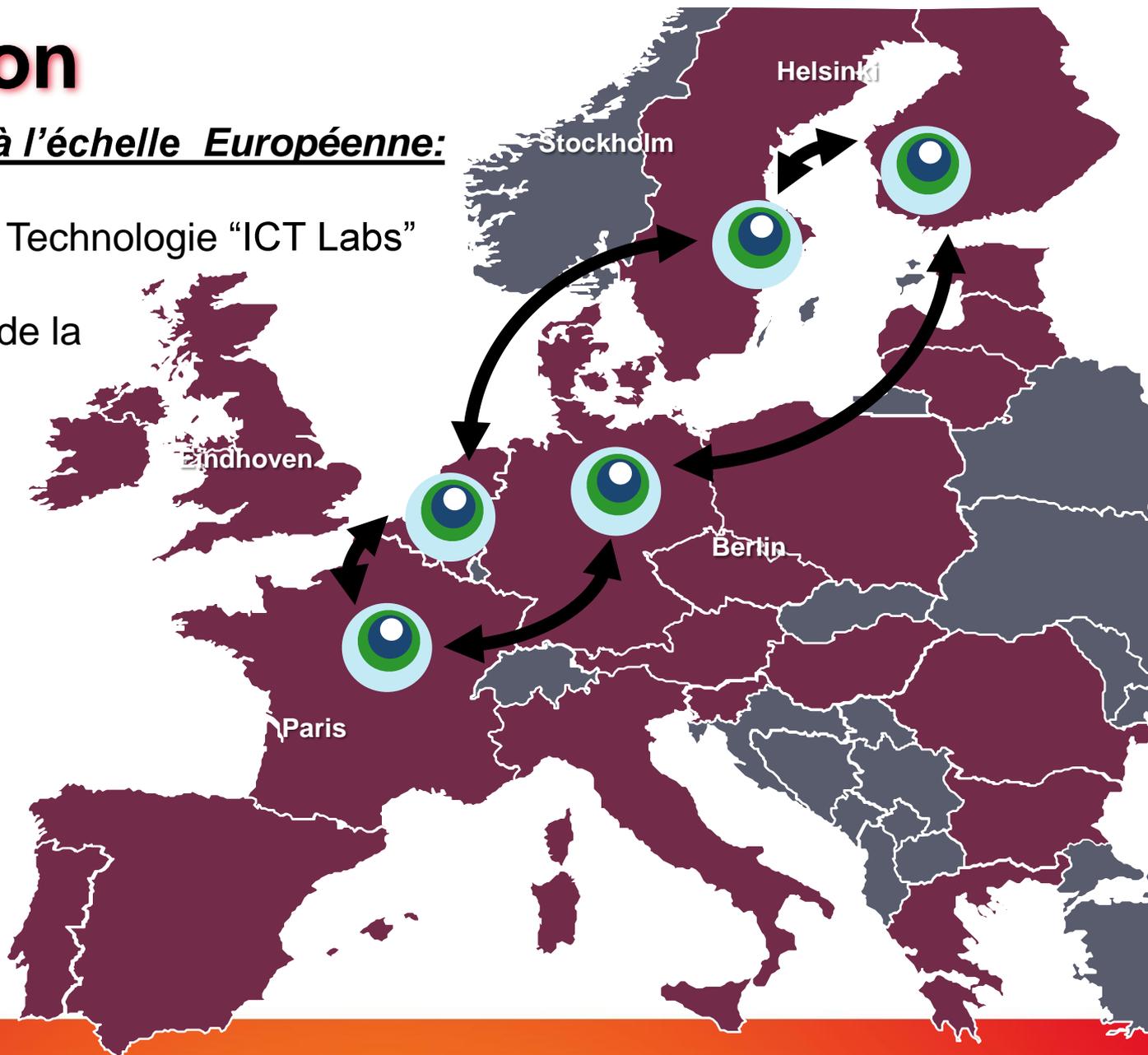
XVIIIème: projets de triangulation à grande échelle

Digression

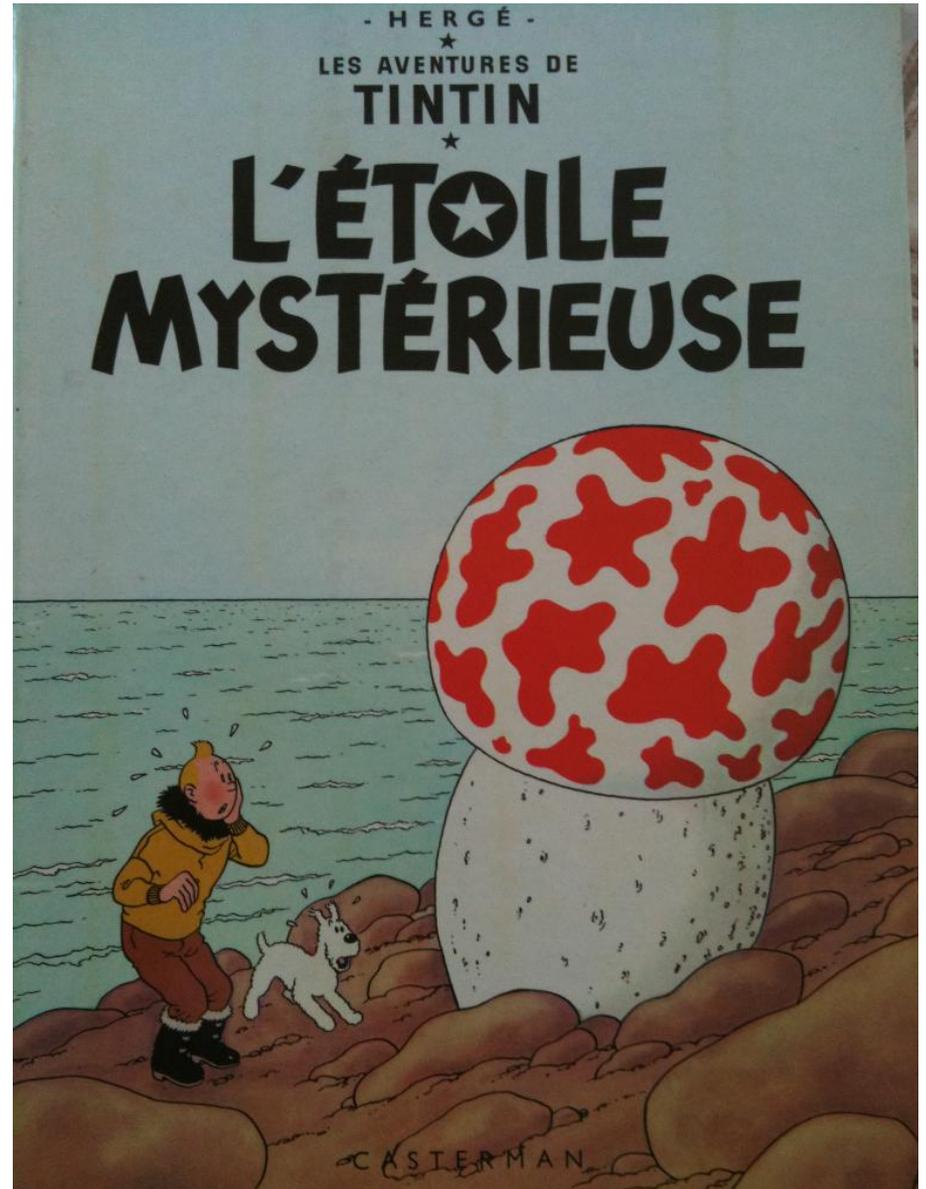
D'autres "graphes" à l'échelle Européenne:

L'institut Européen de Technologie "ICT Labs"

Le Conseil Européen de la Recherche (ERC)

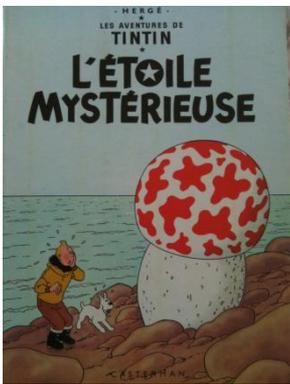


Digression



Digression





Publié en 1941 !!!

Digression



1950: NSF



1984: ESPRIT (FP1 : First European Frame Program)



2006: FP6



FP7 and European Research Council



1. Introduction



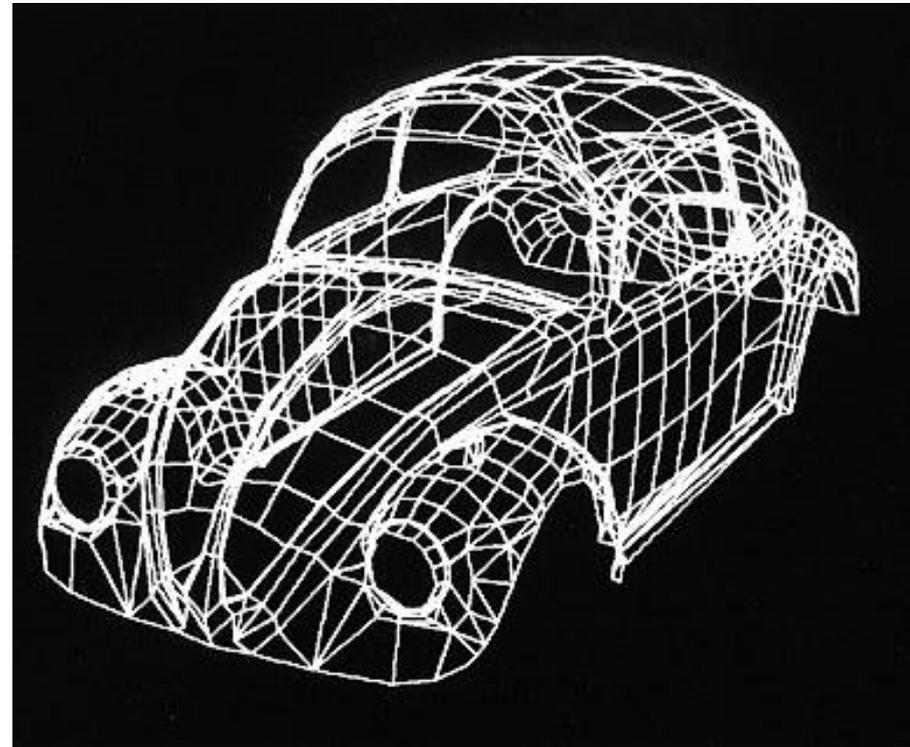
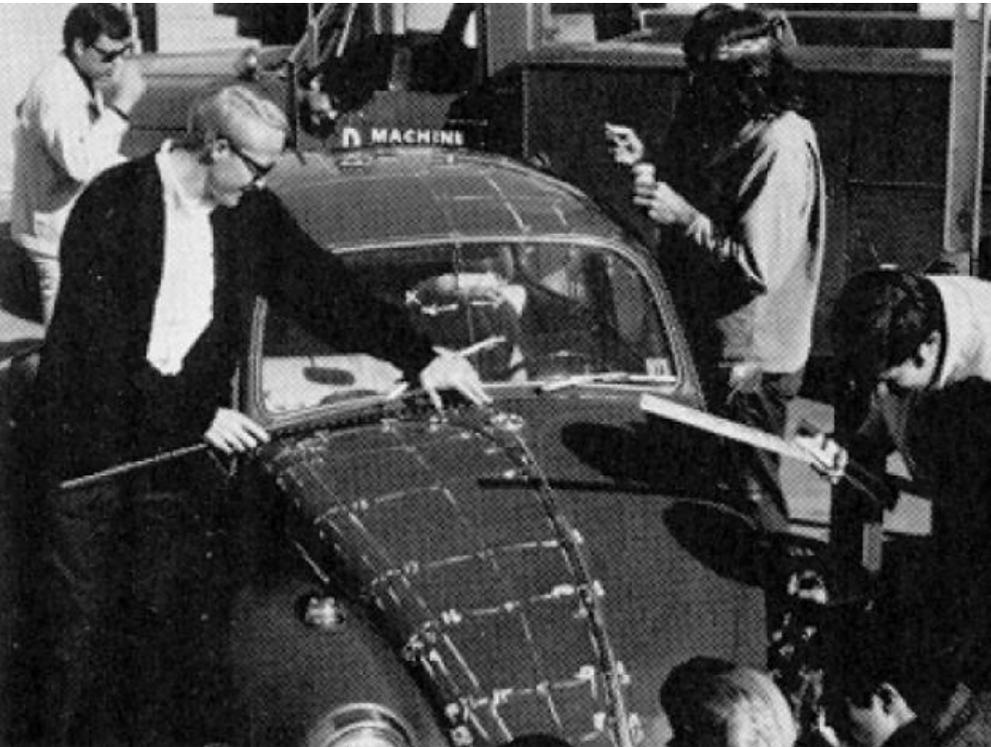
1969 – Henri Gouraud

1. Introduction



1969 – Henri Gouraud

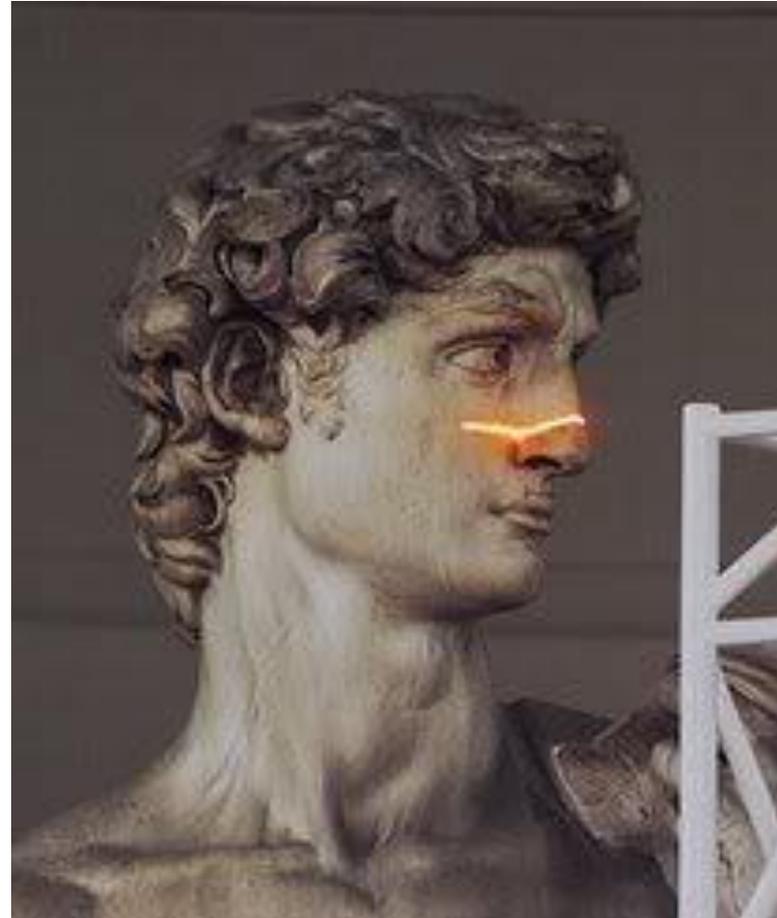
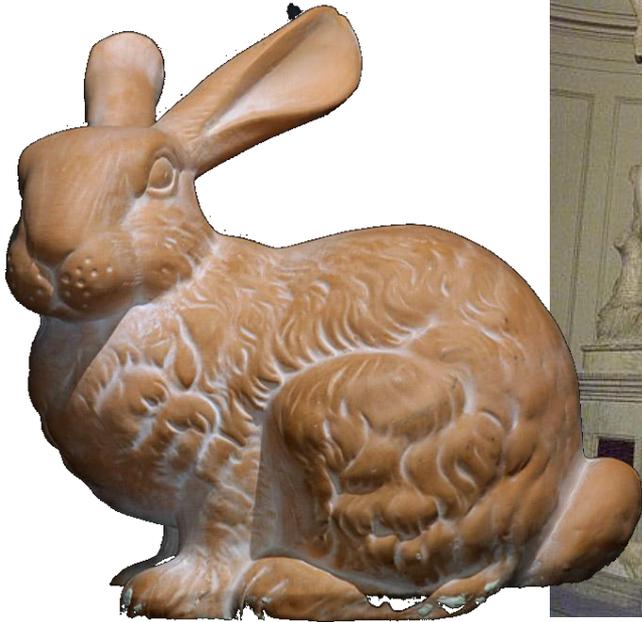
1. Introduction



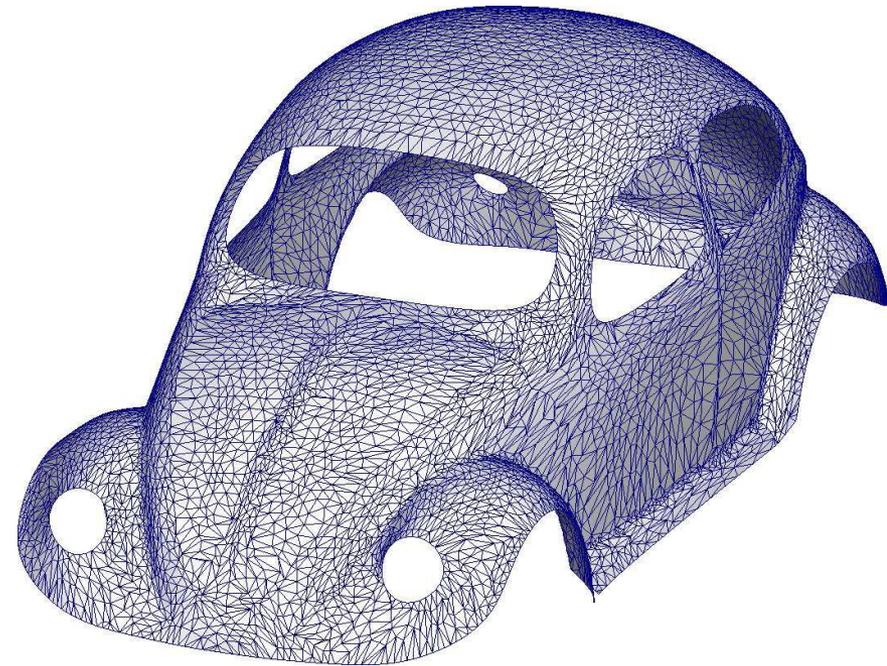
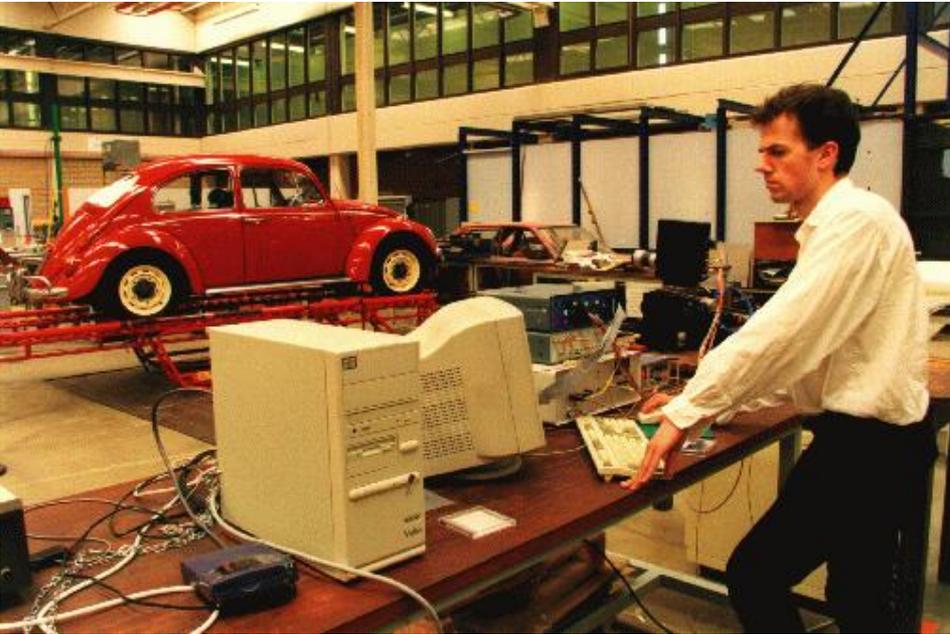
1970 – Ivan Sutherland “digitalise” sa voiture

1. Introduction

Levoy & Turk, Stanford 1994 1998-1999



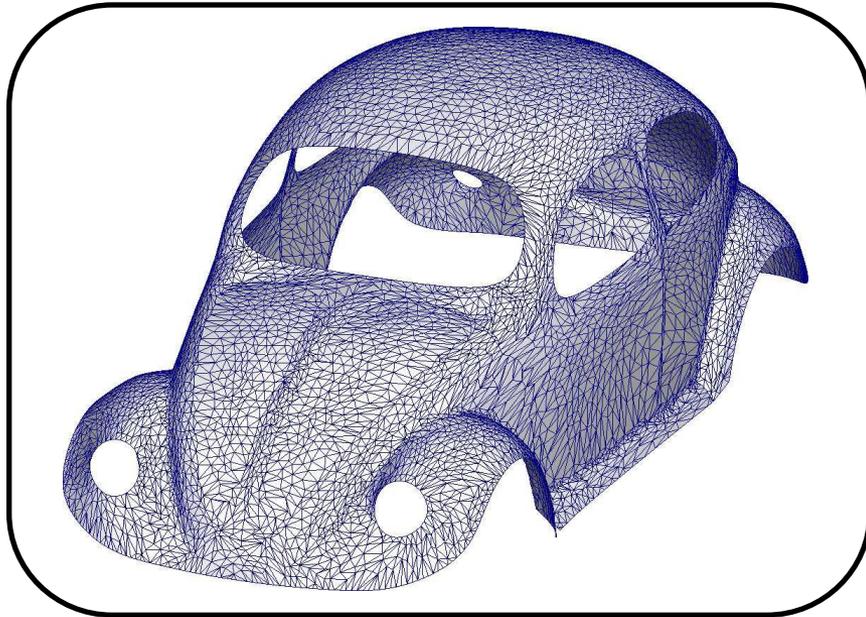
1. Introduction



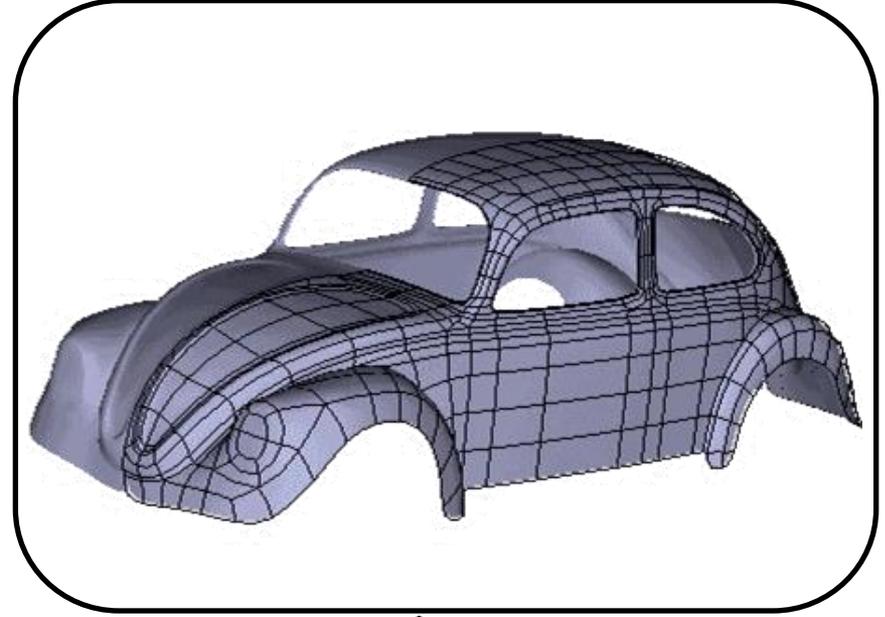
Acquisition 3D d'une voiture

1. Introduction

Maillage issu du scanner



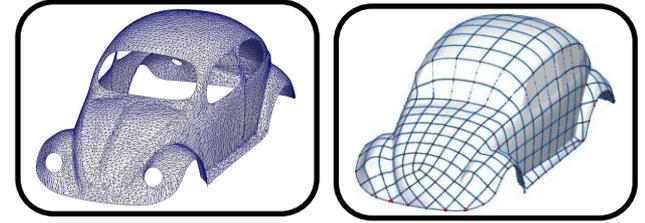
Représentation CAO/CFAO



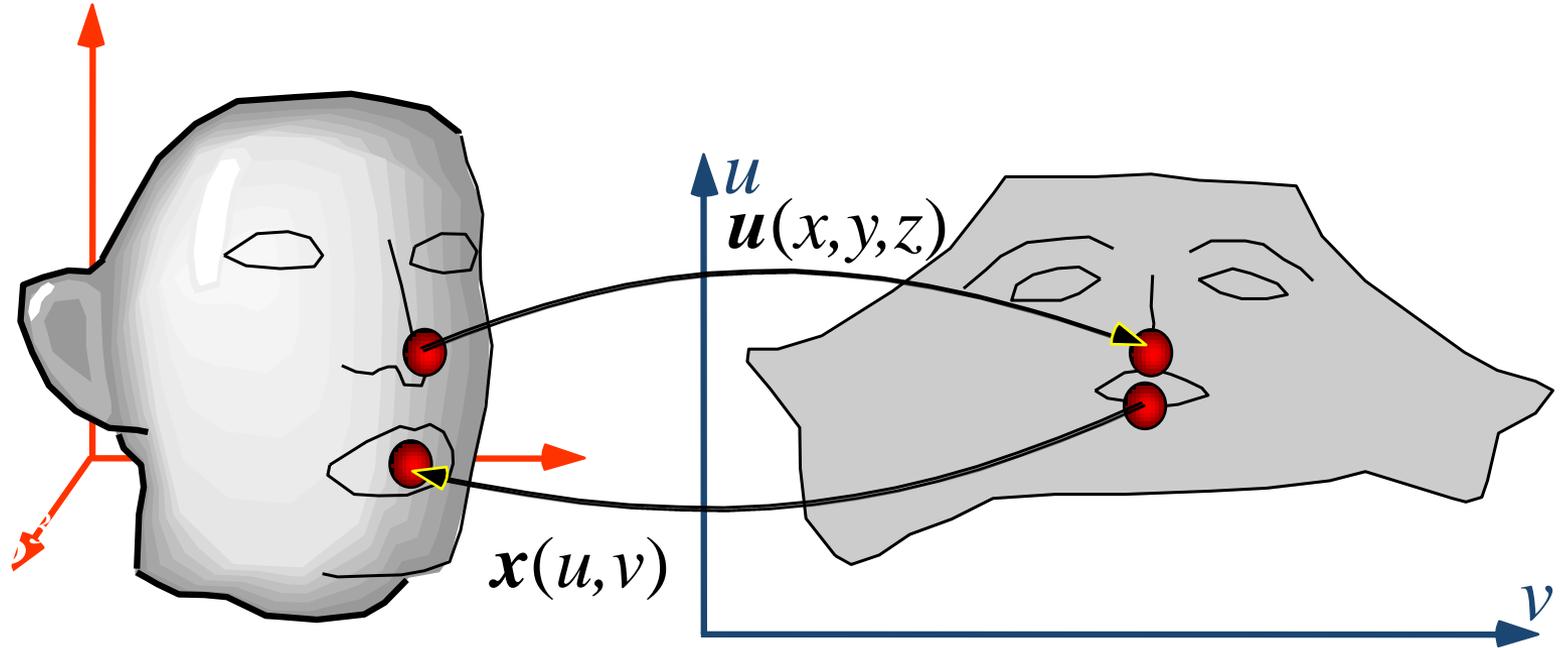
1 million points \rightarrow 100 « Splines » (polynomes de degré 3)

Q: Comment « trouver l'équation » de cette voiture?

2. Approche analytique

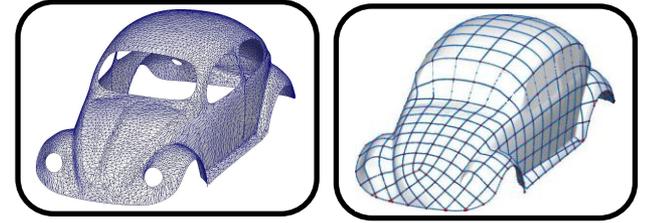


Notion de paramétrisation

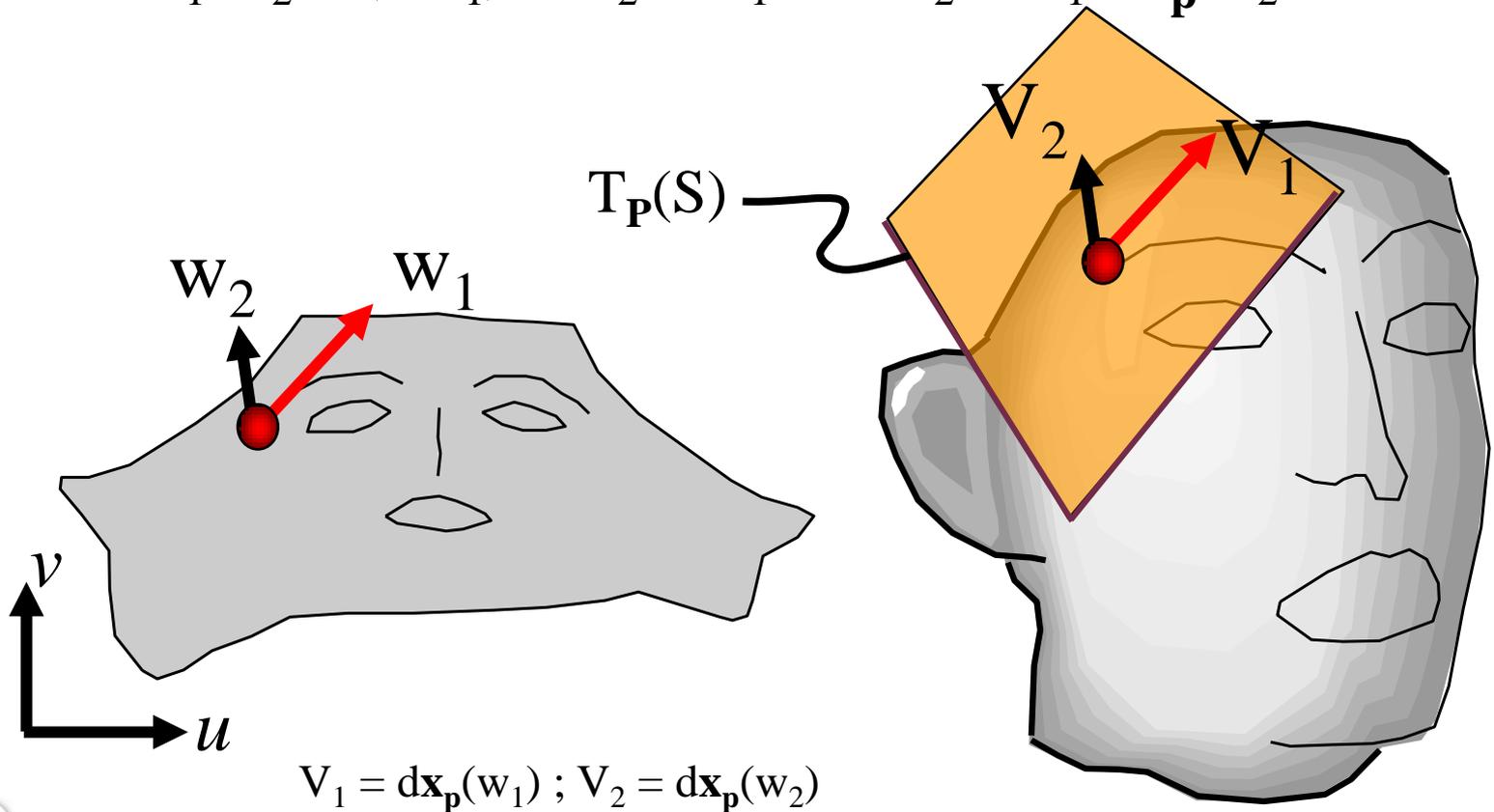


2. Approche analytique

Géométrie Riemannienne

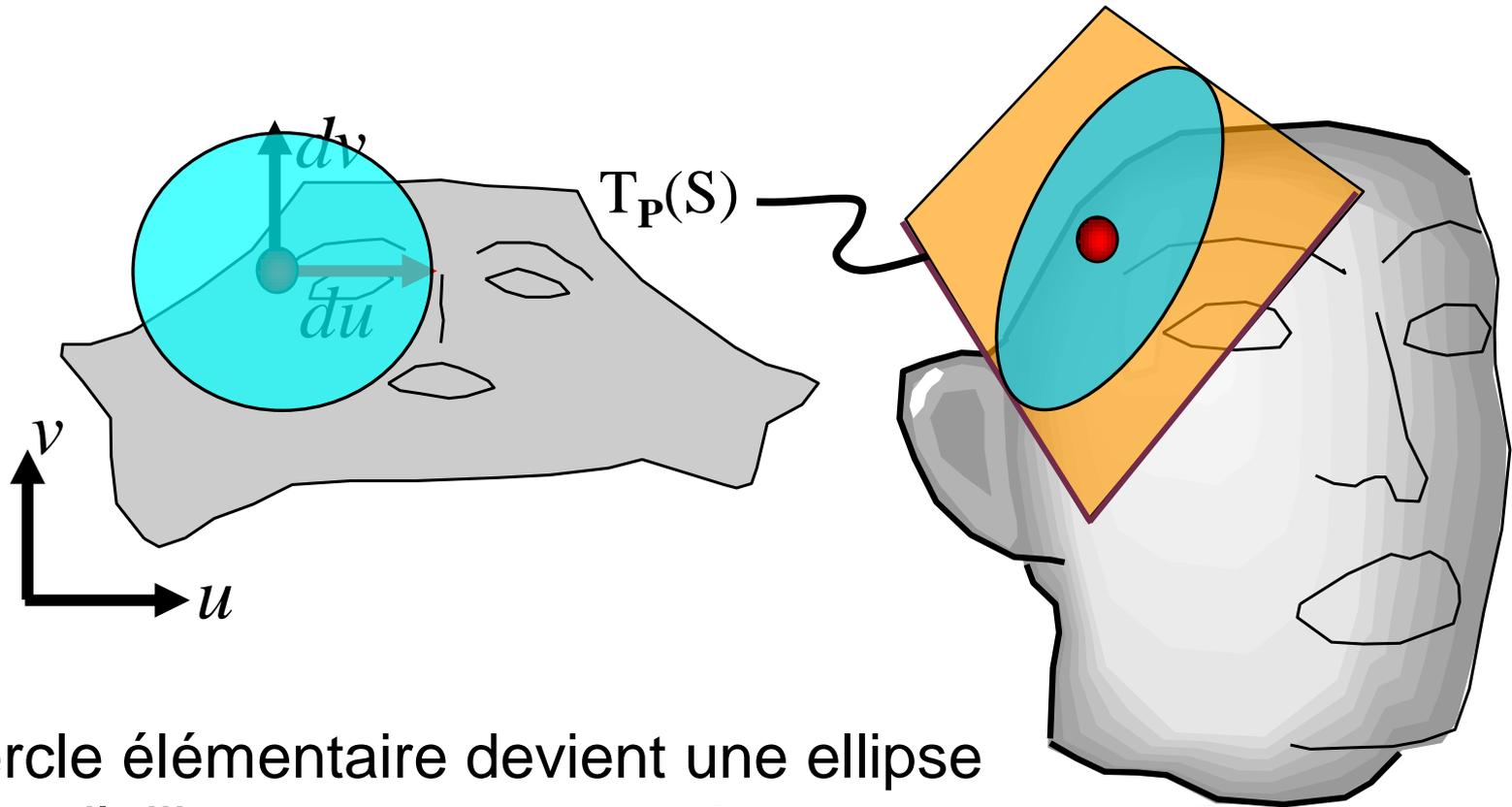
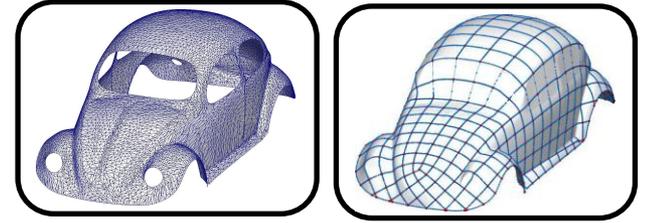


$$V_1^t V_2 = (J w_1)^t J w_2 = w_1^t J^t J w_2 = w_1^t \mathbf{G}_p w_2$$



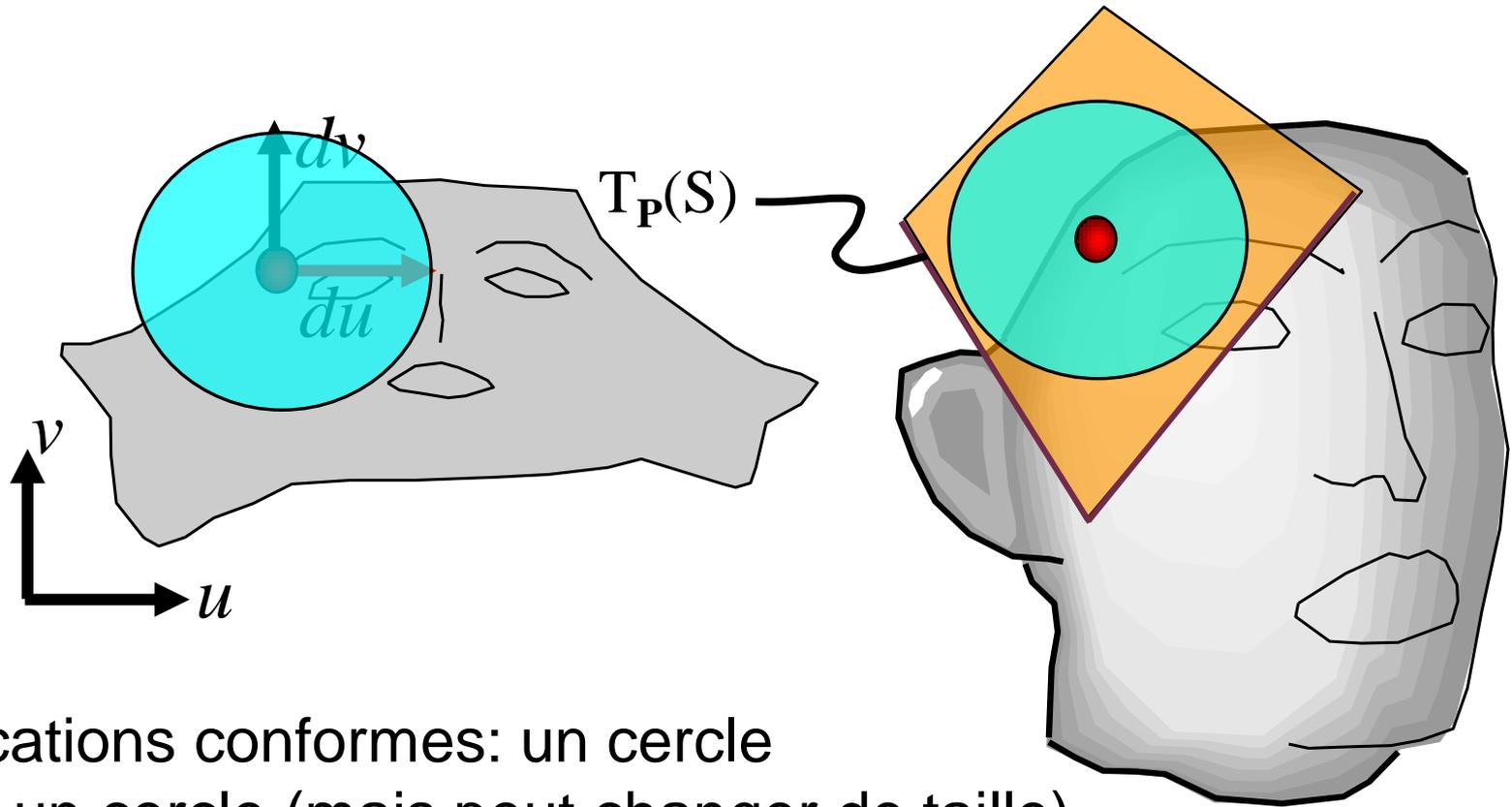
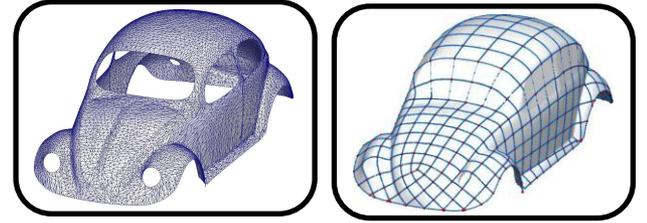
2. Approche analytique

Géométrie Riemannienne



Un cercle élémentaire devient une ellipse
Axes de l'ellipse: vecteurs et valeurs propres de \mathbf{G}

2. Approche analytique Géométrie Riemannienne

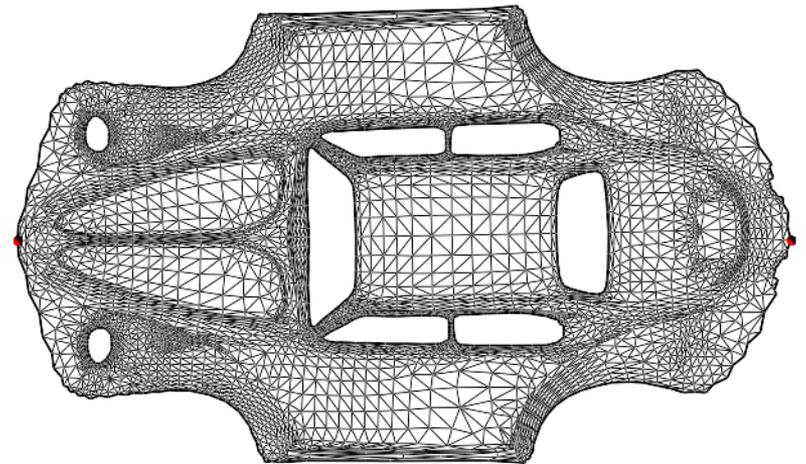
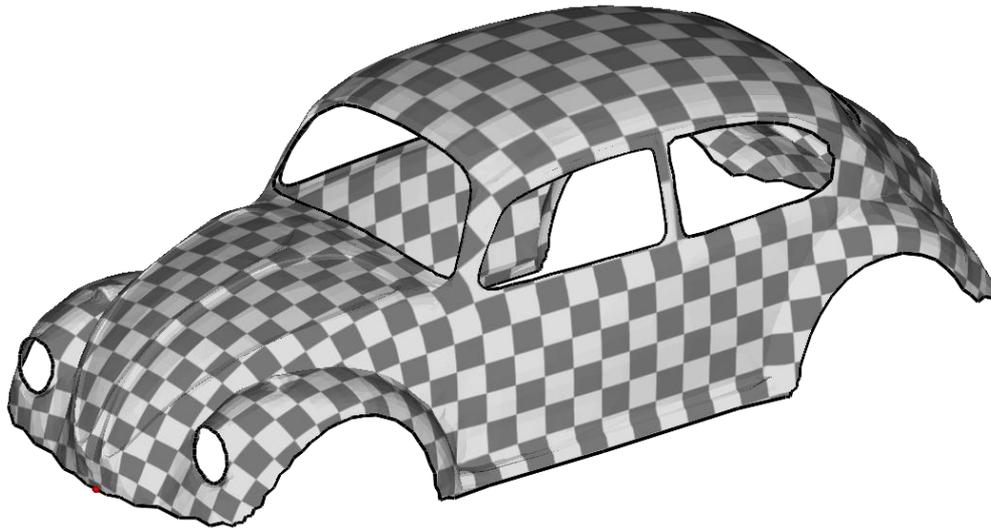


Applications conformes: un cercle
reste un cercle (mais peut changer de taille)

2. Approche analytique Least Squares Conformal Maps

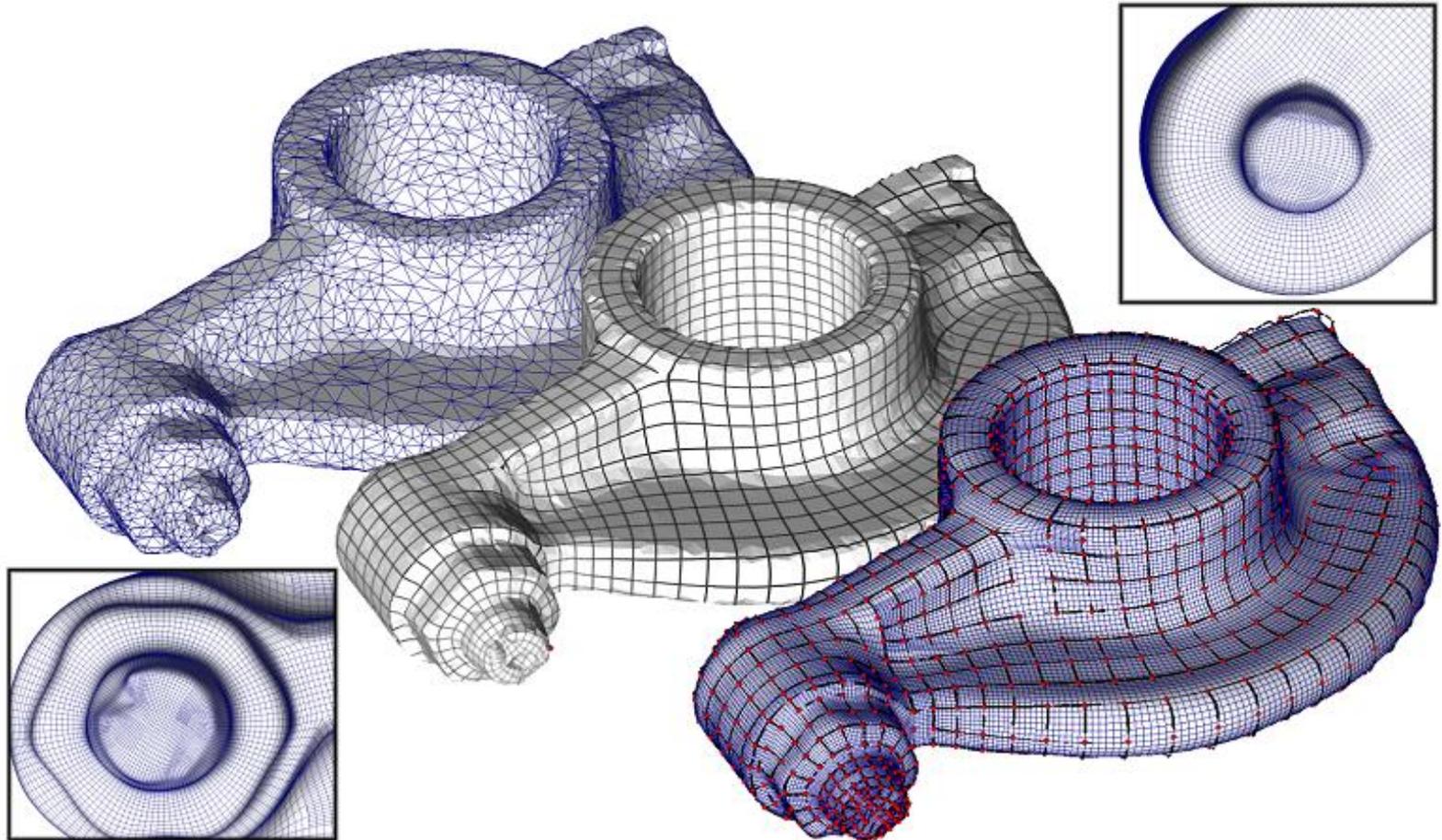
Minimiser $\sum_T \left\| \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} - \begin{bmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{bmatrix} \right\|^2$

[L, Petitjean, Ray Maillot 2002]



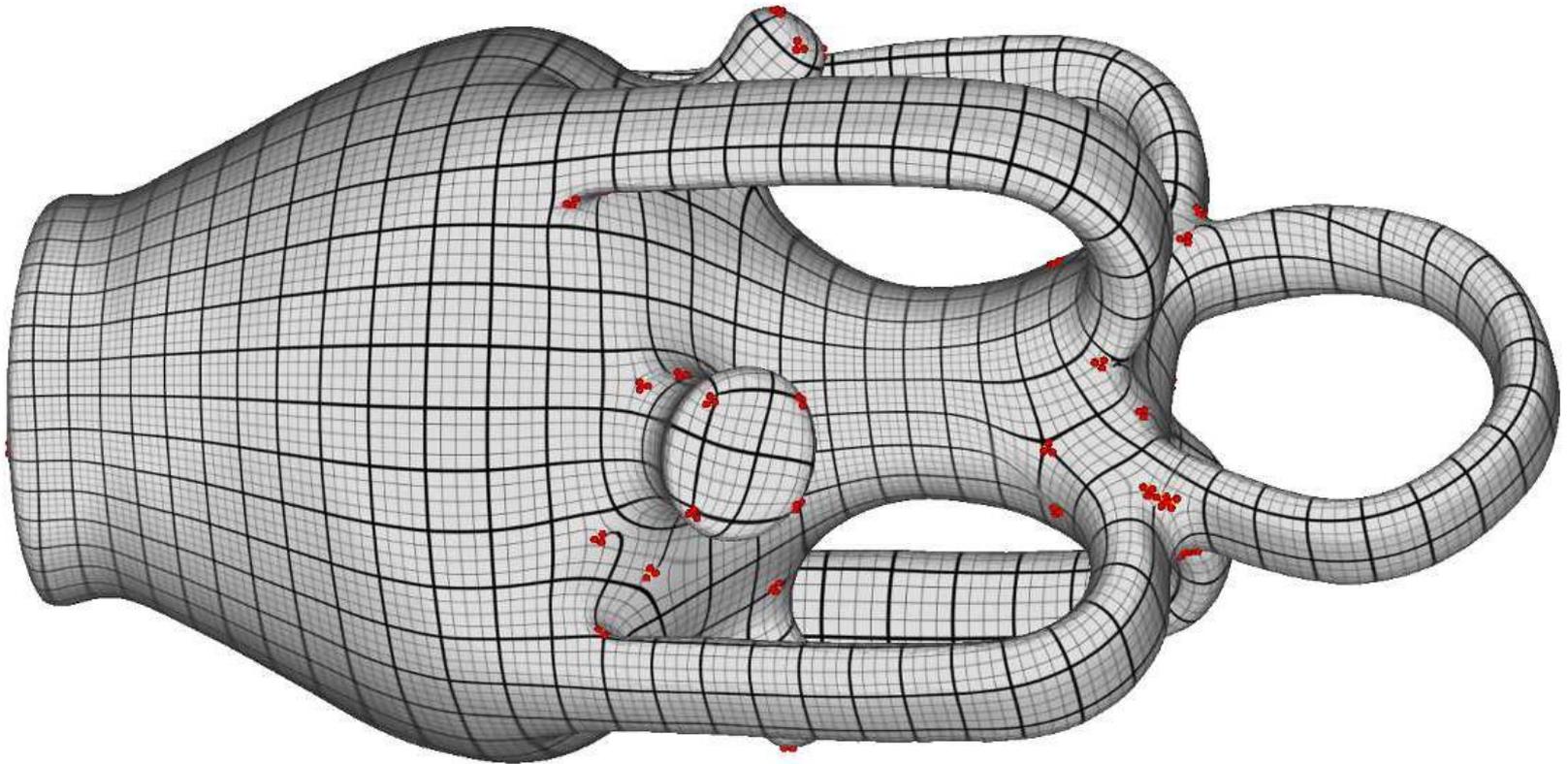
2. Approche analytique

Paramétrisation Globale Périodique [Ray, Li, L, Sheffer, Alliez 2008]



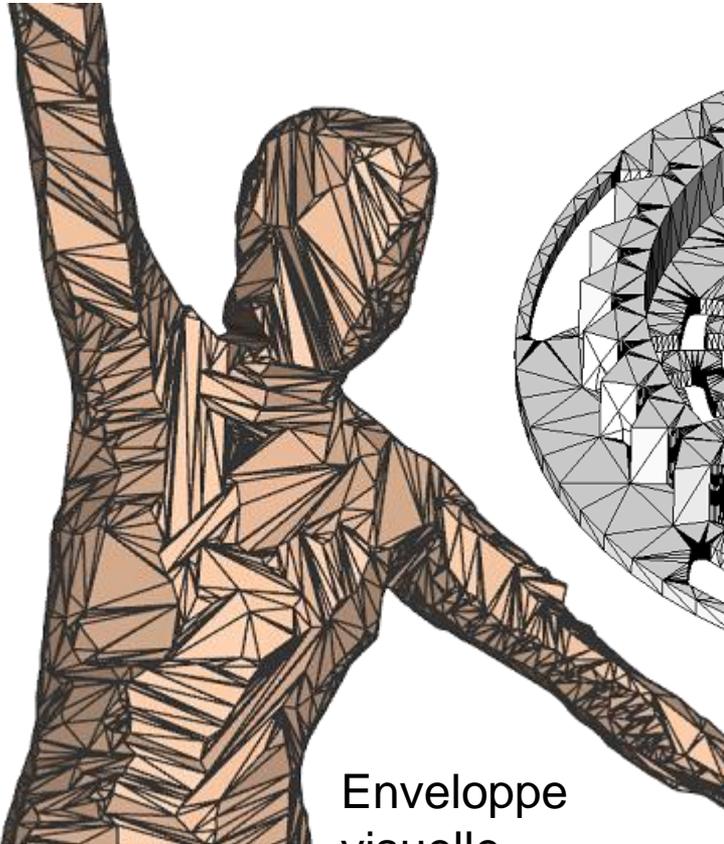
2. Approche analytique

Paramétrisation Globale Périodique [Ray, Li, L, Sheffer, Alliez 2008]

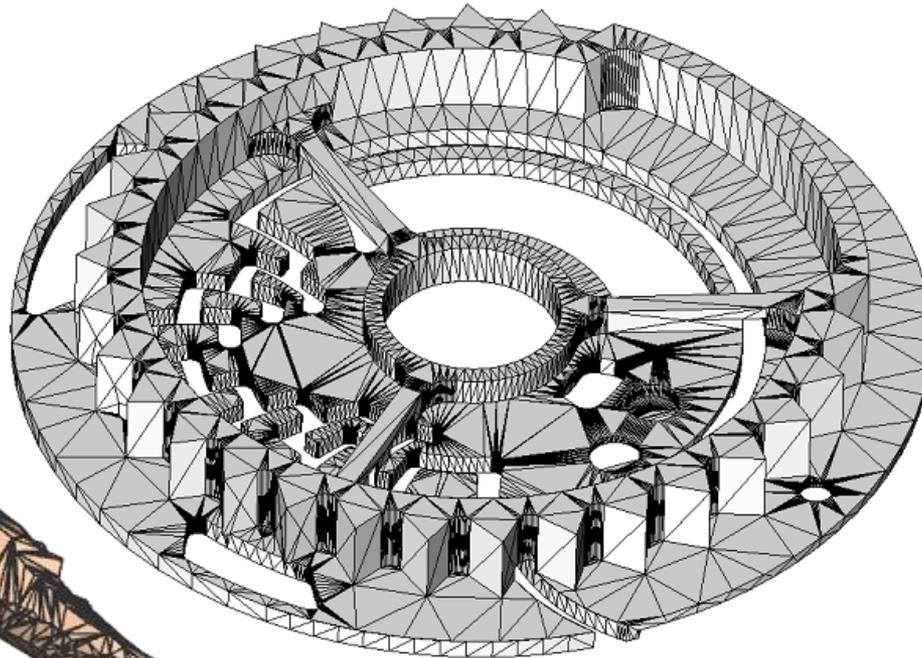


2. Approche analytique

Limites de l'approche analytique



Enveloppe
visuelle

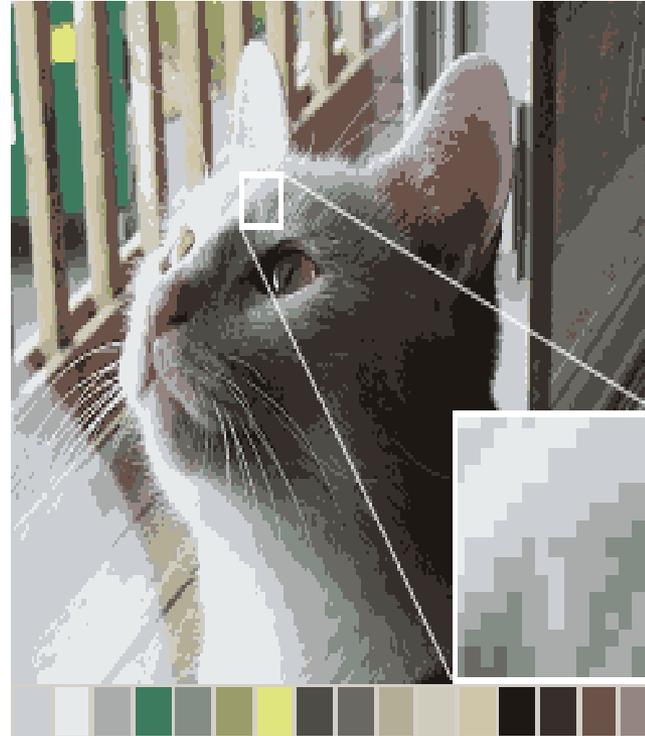
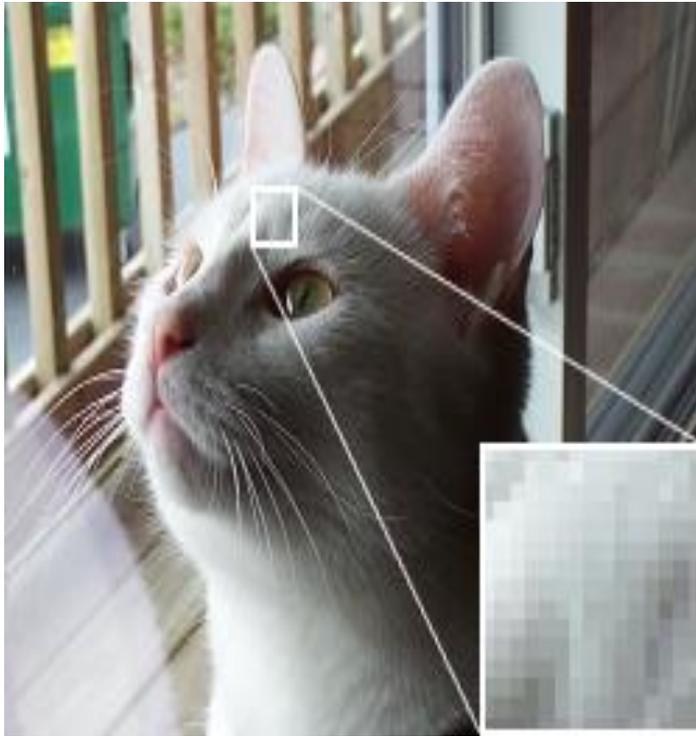


Maillage issue de la CAO

Triangles
« aplatis »



3. Approche par l'échantillonnage

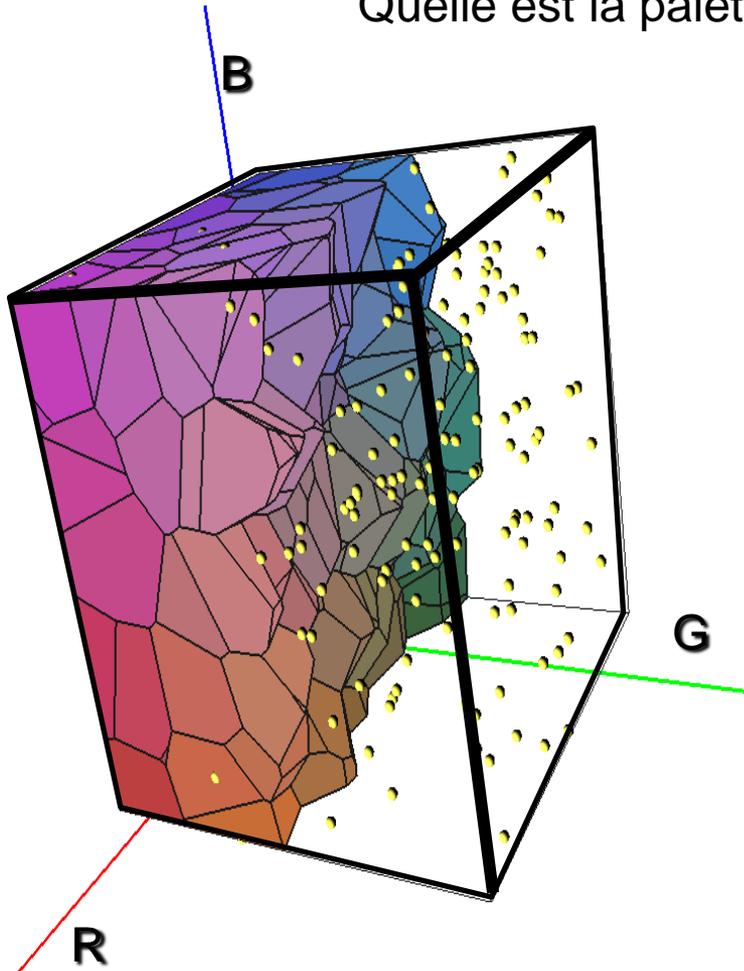


Quantisation des couleurs

[Leung et.al, GPU Pro, AK Peters, 2010]

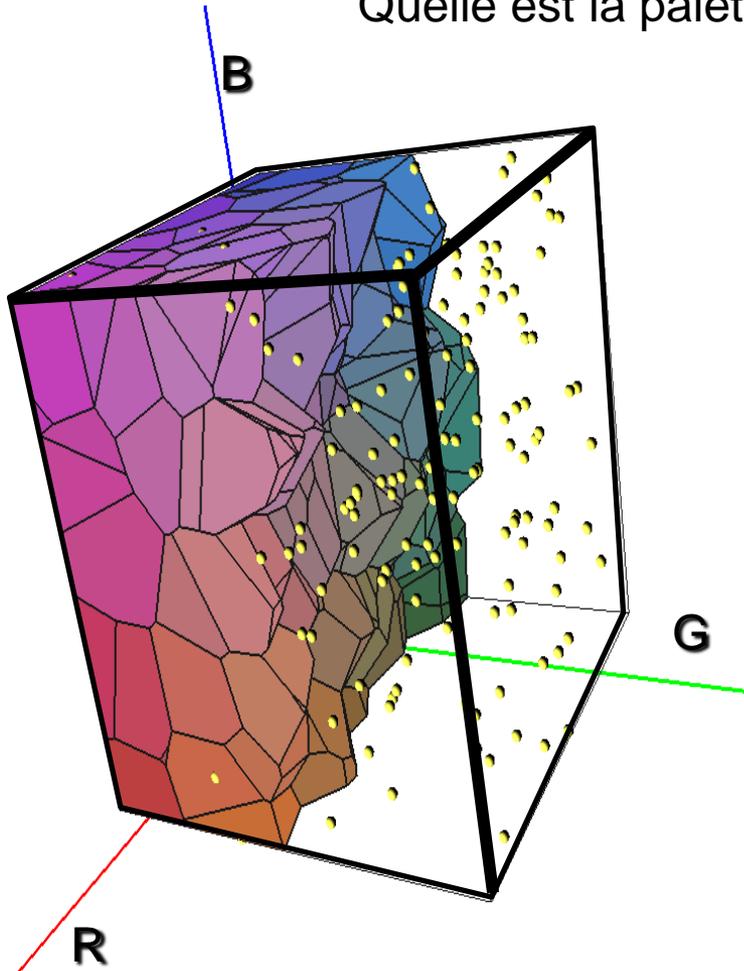
3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?



3. Approche par l'échantillonnage

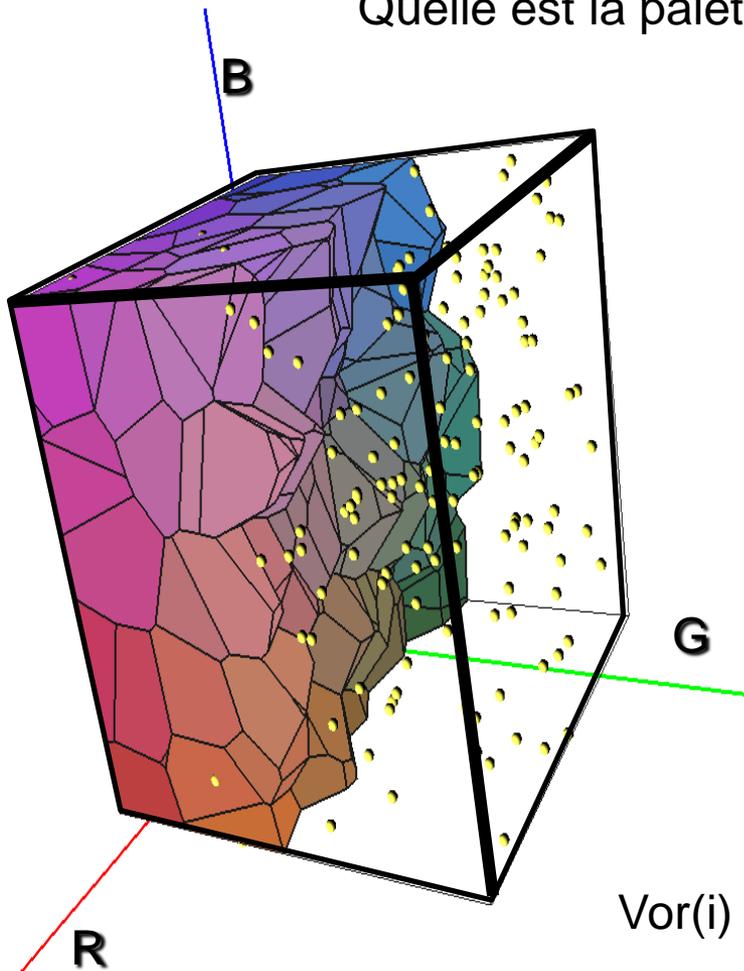
Quelle est la palette de couleurs optimale ?



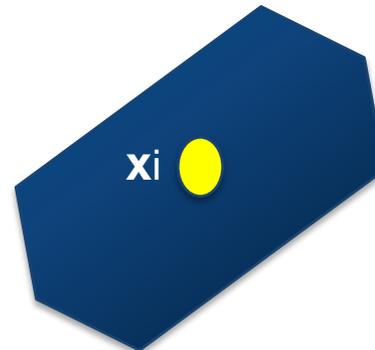
$x_i = (r_i, g_i, b_i)$ Entrée de la palette

3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?



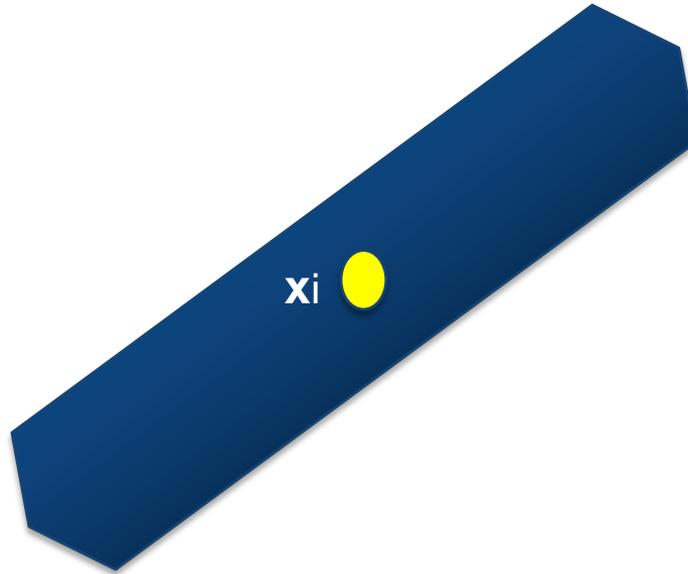
$\mathbf{x}_i = (r_i, g_i, b_i)$ Entrée de la palette



$$\text{Vor}(i) = \{ \mathbf{x} / d(\mathbf{x}, \mathbf{x}_i) < d(\mathbf{x}, \mathbf{x}_j) \} \forall i \neq j$$

3. Approche par l'échantillonnage

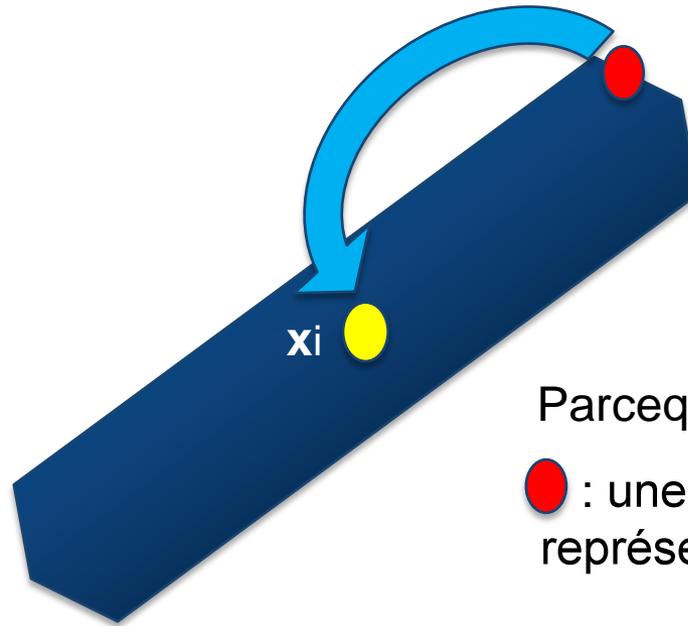
Quelle est la palette de couleurs optimale ?



Une « mauvaise » entrée de la palette / cellule de Voronoi

3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?



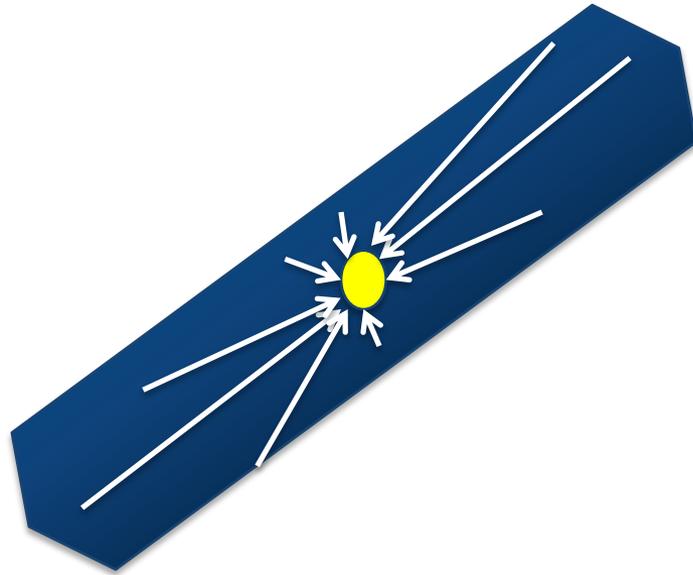
Pourquoi mauvaise ?

Parceque Vor(xi) contient
● : une couleur qui n'est pas bien
représentée par l'entrée xi ●

Une « mauvaise » entrée de la palette / cellule de Voronoi

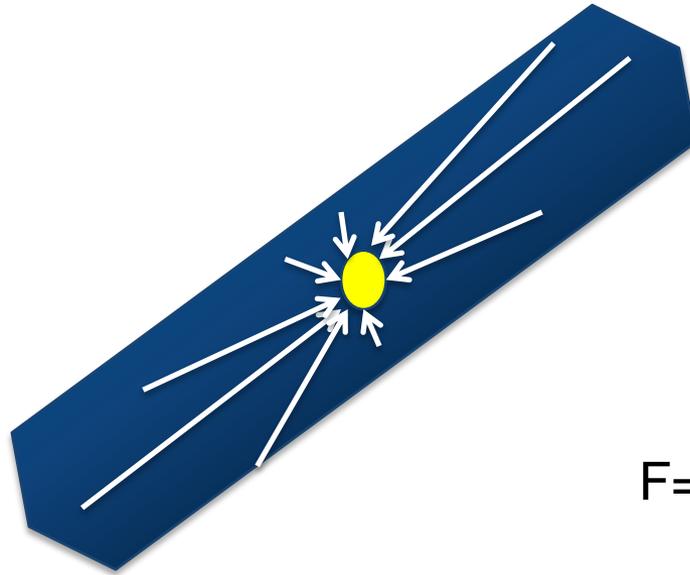
3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?



3. Approche par l'échantillonnage

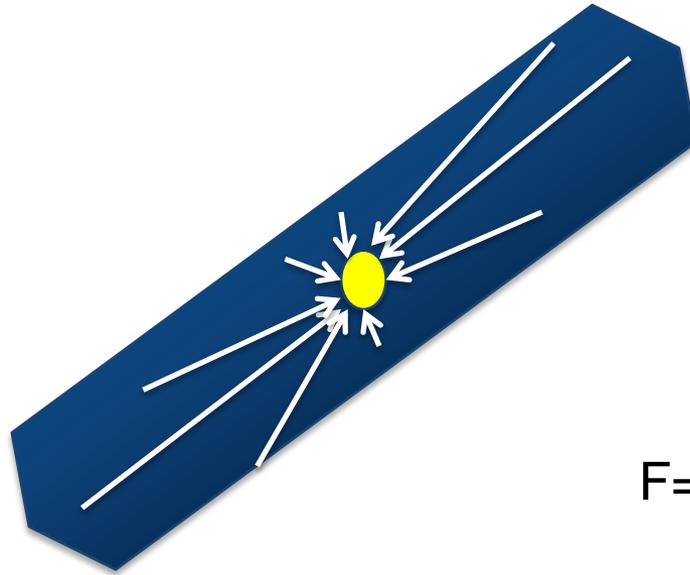
Quelle est la palette de couleurs optimale ?



$$F = \int_{\text{Vor}(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 dx$$

3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?



$$F = \int_{\text{Vor}(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 dx$$

F: Puissance du bruit de l'échantillonnage

3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?

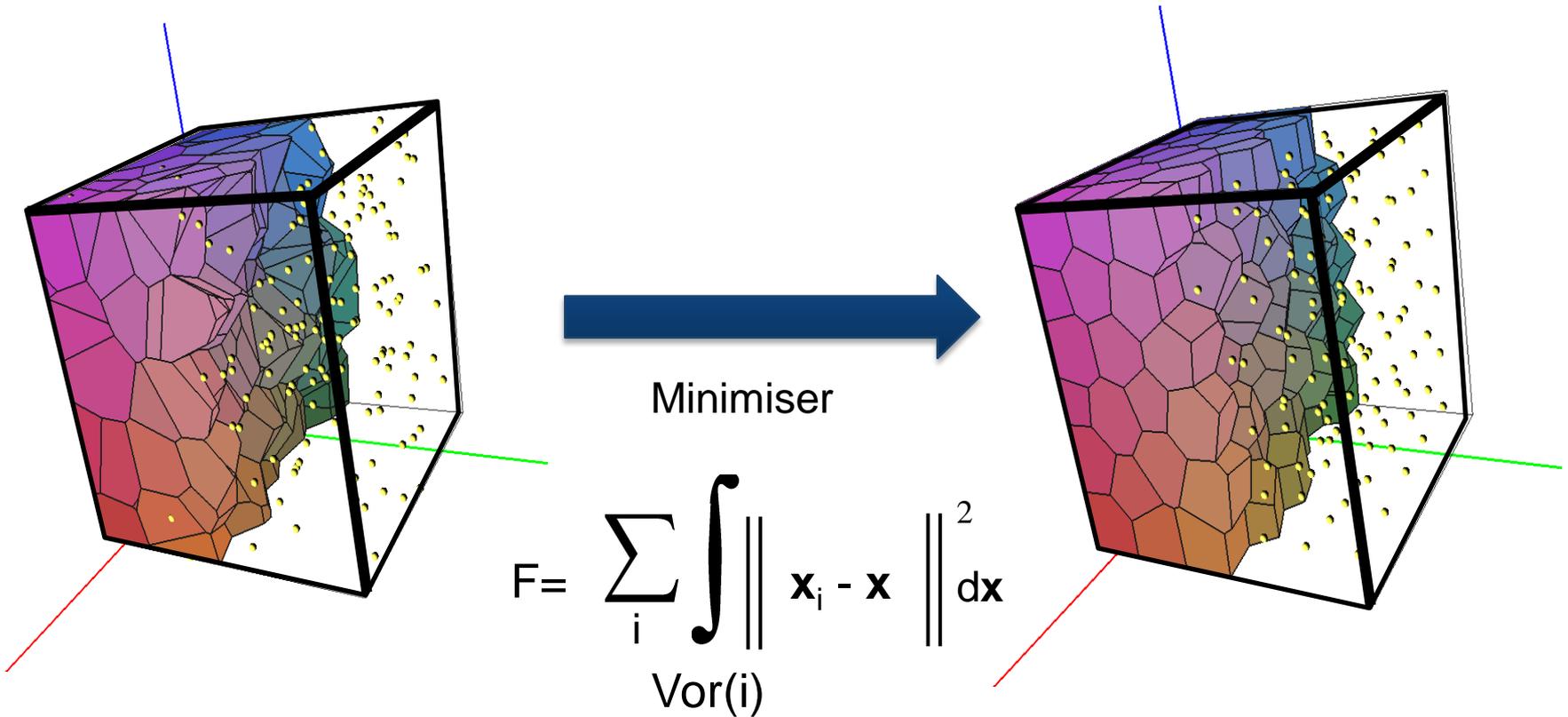


$$F = \int_{\text{Vor}(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 dx$$

F: Puissance du bruit de l'échantillonnage

3. Approche par l'échantillonnage

Quelle est la palette de couleurs optimale ?



3. Approche par l'échantillonnage

La méthode classique

Relaxation de Lloyd = descente de gradient

$$F = \sum_i \int_{\text{Vor}(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 dx$$

$$\nabla F |_{\mathbf{x}_i} = 2 m_i (\mathbf{x}_i - \mathbf{g}_i)$$

[Iri et.al], [Du et.al]

Volume de Vor(i)

Barycentre de Vor(i)

3. Approche par l'échantillonnage

La méthode classique

Relaxation de Lloyd = descente de gradient

$$F = \sum_i \int_{\text{Vor}(i)} \| \mathbf{x}_i - \mathbf{x} \|^2 dx$$

$$\nabla F|_{\mathbf{x}_i} = 2 m_i (\mathbf{x}_i - \mathbf{g}_i) \quad \begin{array}{l} [\text{Iri et.al}], \\ [\text{Du et.al}] \end{array}$$

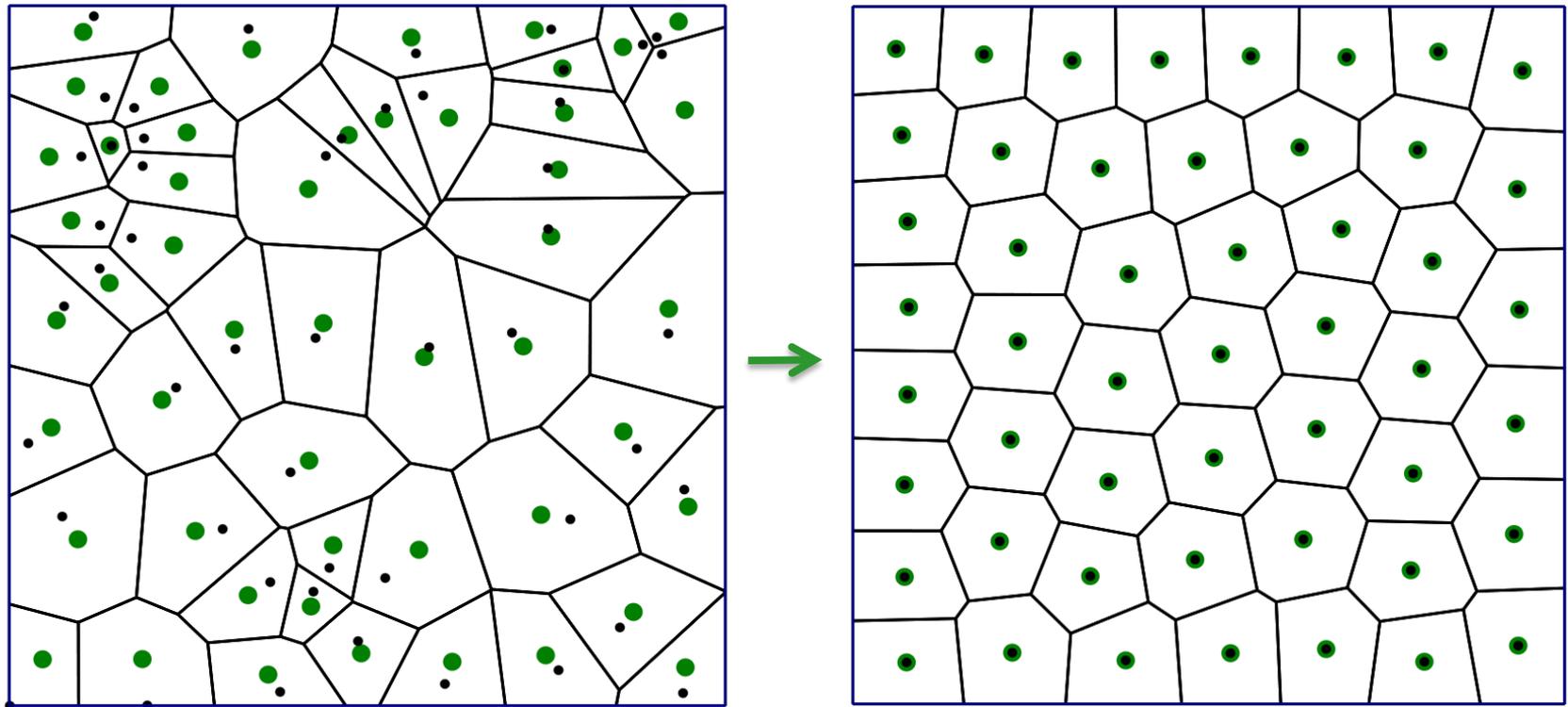
Volume de Vor(i)

Barycentre de Vor(i)

Si \mathbf{x}_i correspond au barycentre de Vor(i), alors on a un point stationnaire de F (et donc un « bon échantillonnage »)

Vor(X): **Diagramme de Voronoi barycentrique**

3. Approche par l'échantillonnage



3. Approche par l'échantillonnage

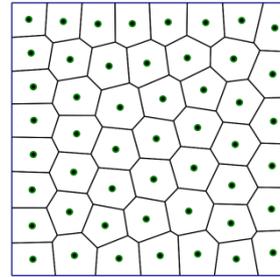
Théorème: F est C^2 presque partout

[Liu, Wang, L, Sun, Yan, Lu and Yang 09]

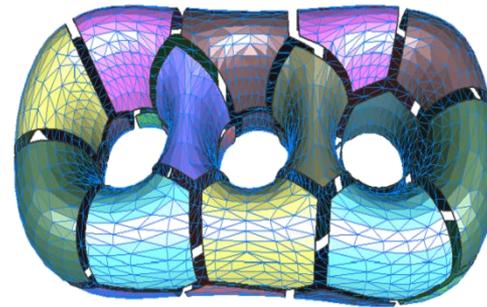
Conséquence: on peut utiliser un algorithme de minimisation bien plus efficace.

3. Approche par l'échantillonnage

CVT en 2D

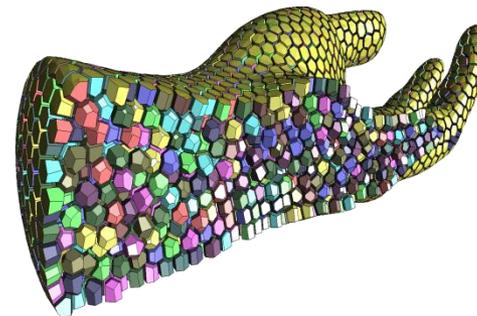


CVT sur surfaces

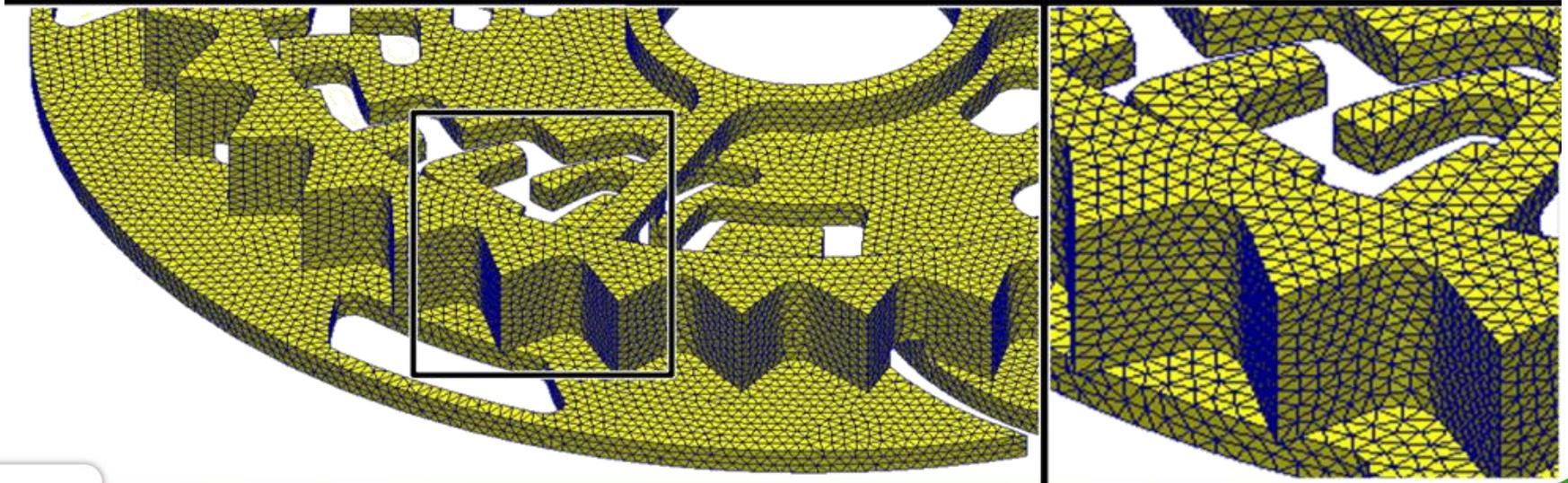
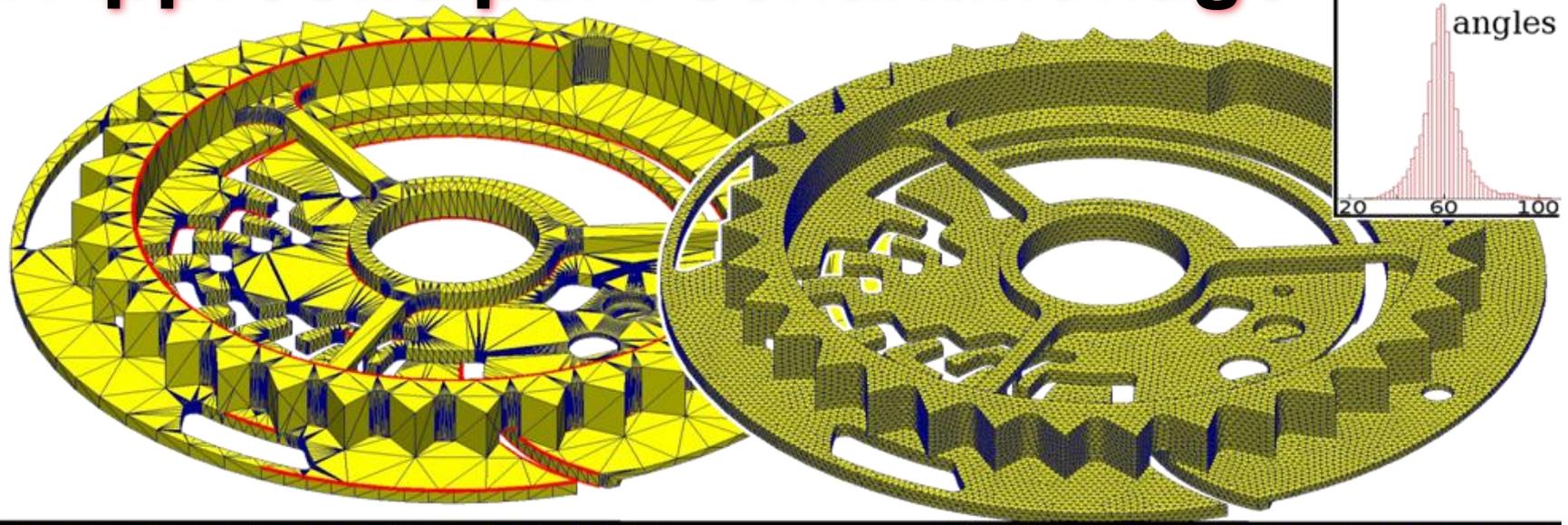


CVT en volumes

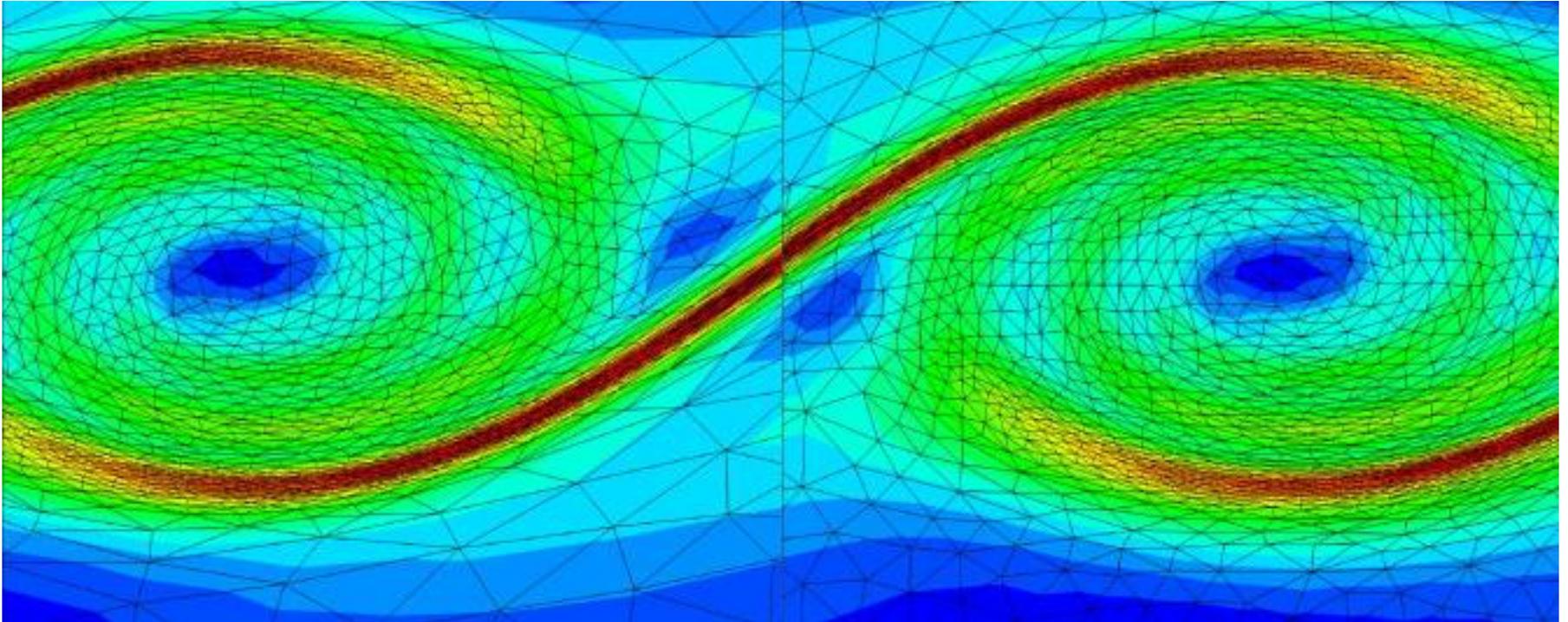
[Yan, Wang, L, Liu SIGGRAPH 2010]



3. Approche par l'échantillonnage



4. Maillages anisotropes

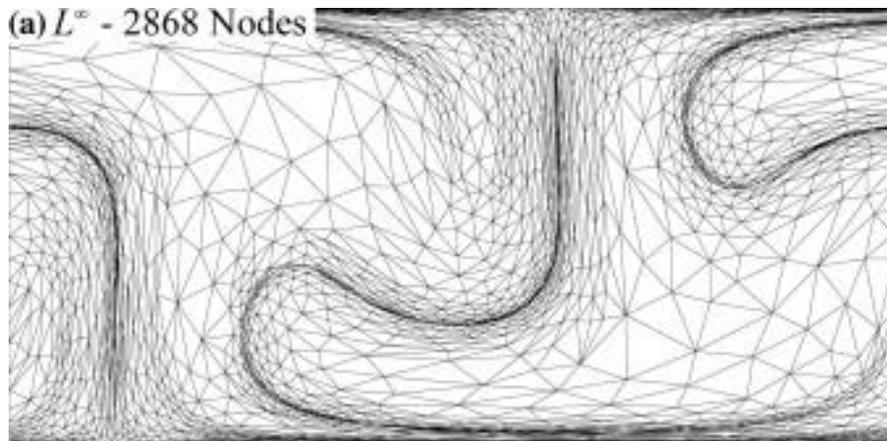


Maillage **anisotrope** adapté à la solution (à la physique)
[Miron et.al, Journal of Computational Physics, 2010]

4. Maillages anisotropes



Il a des triangles allongés là où on en a besoin (à savoir, qui suivent la physique)

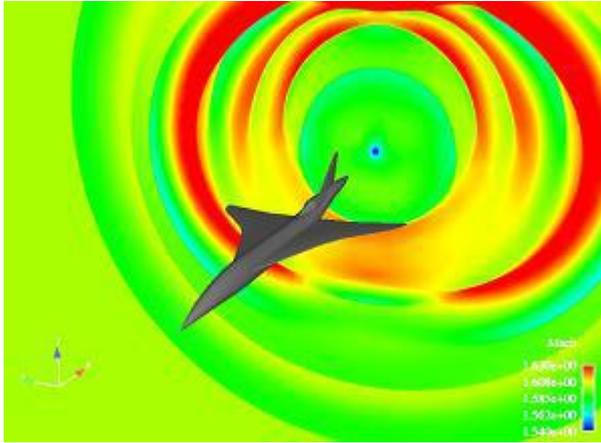


Bénéfice: plus de précision avec **moins** de triangles

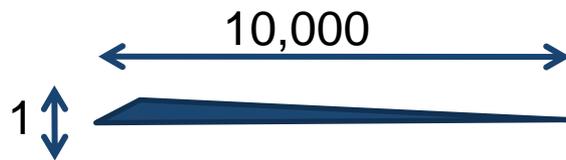
Triangles allongés: pas toujours “mauvais”, parfois on les veut, mais **contrôler** leur forme, taille et orientation.

[Miron et.al, Journal of Computational Physics, 2010]

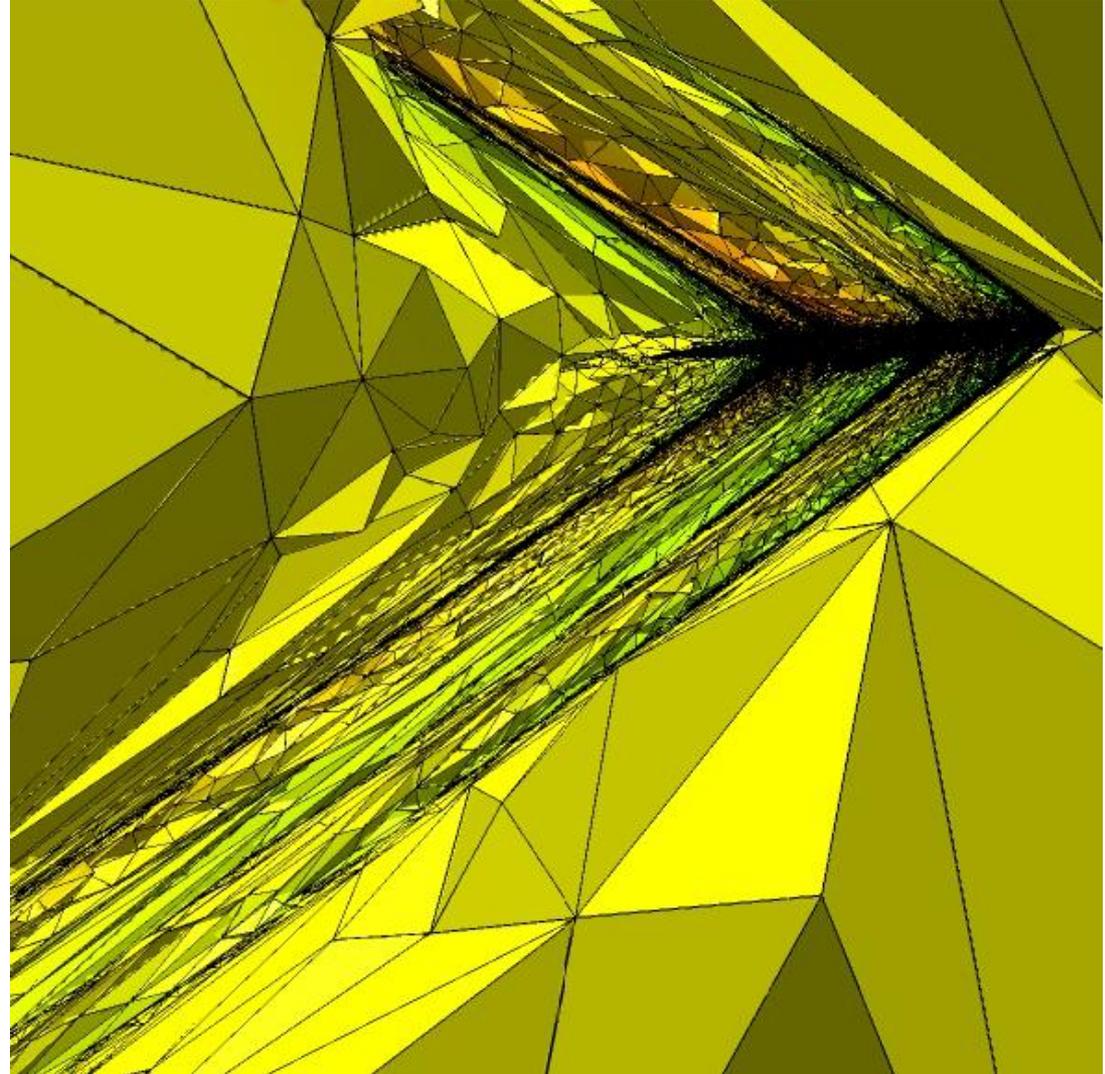
4. Maillages anisotropes



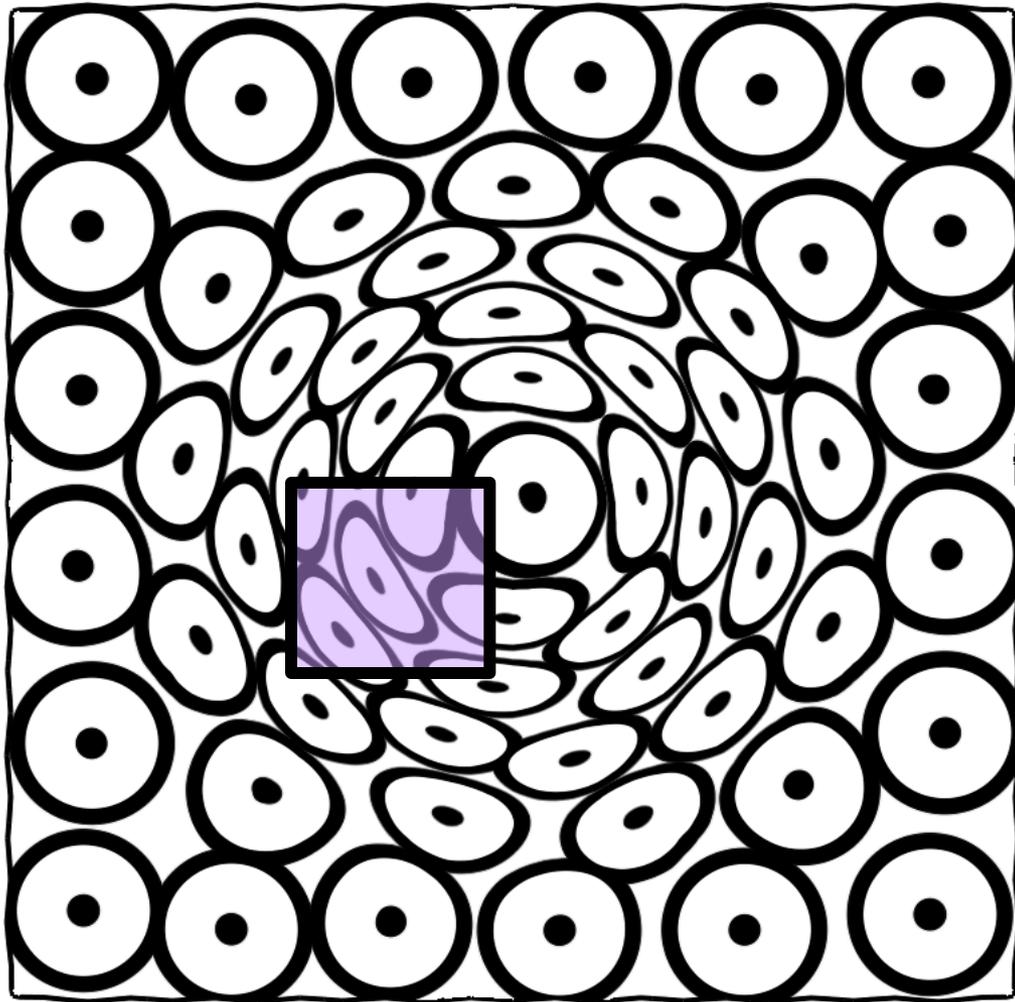
Vol supersonique
[Alauzet et.al]



Rapport d'aspect:
1:10,000 (typique)



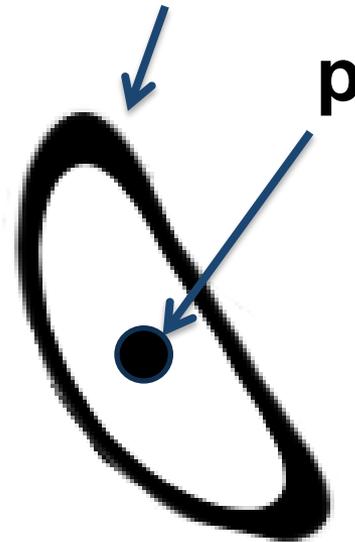
4. Maillages anisotropes



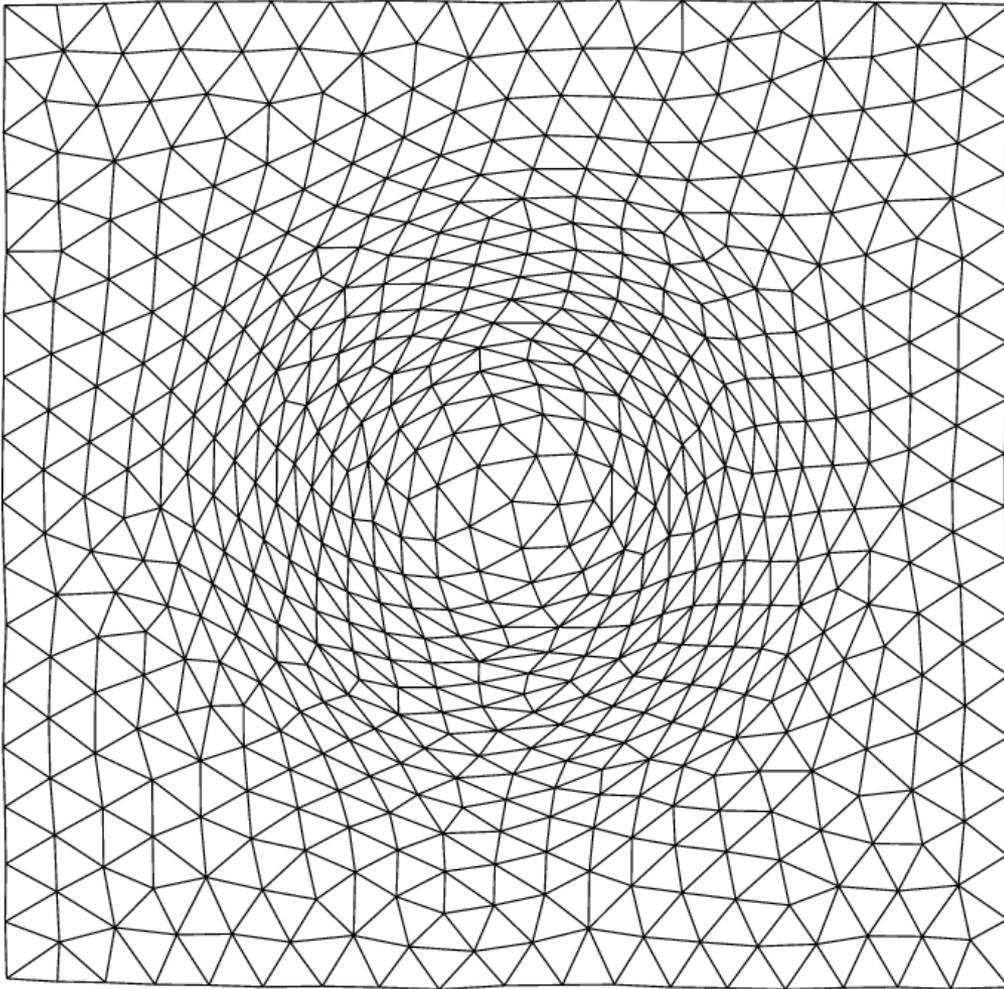
En entrée: une métrique

$$G(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

$$\{ \mathbf{q} \mid d_G(\mathbf{p}, \mathbf{q}) = 1 \}$$

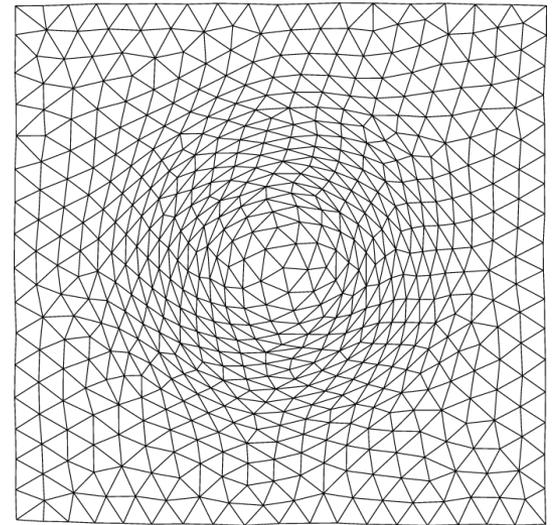
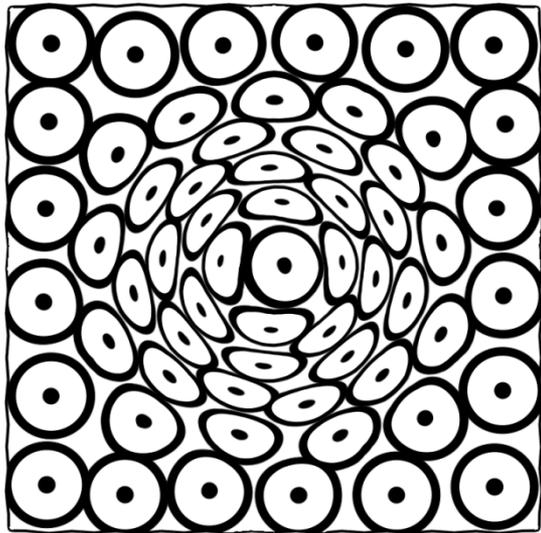
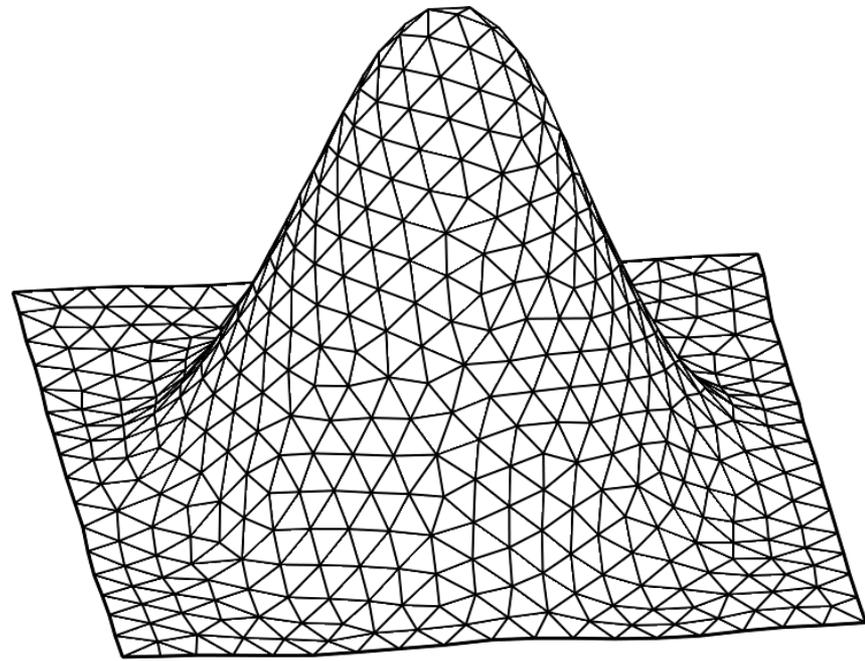
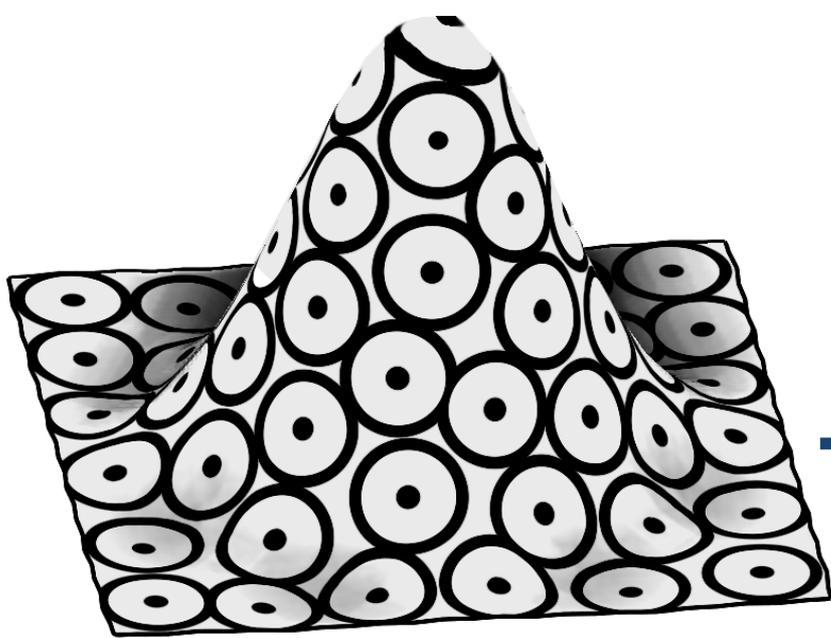


4. Maillages anisotropes



Le résultat: triangles déformés par l'anisotropie

Q: Comment calculer un diagramme de Voronoi barycentrique anisotrope ?



4. Maillages anisotropes

L'idée

Remplace l'**anisotropie** par des **dimensions en plus**

Note: Parfois, on va avoir besoin de plus de 3 dimensions

Il va en falloir combien ?

Théorème du plongement isométrique de John Nash

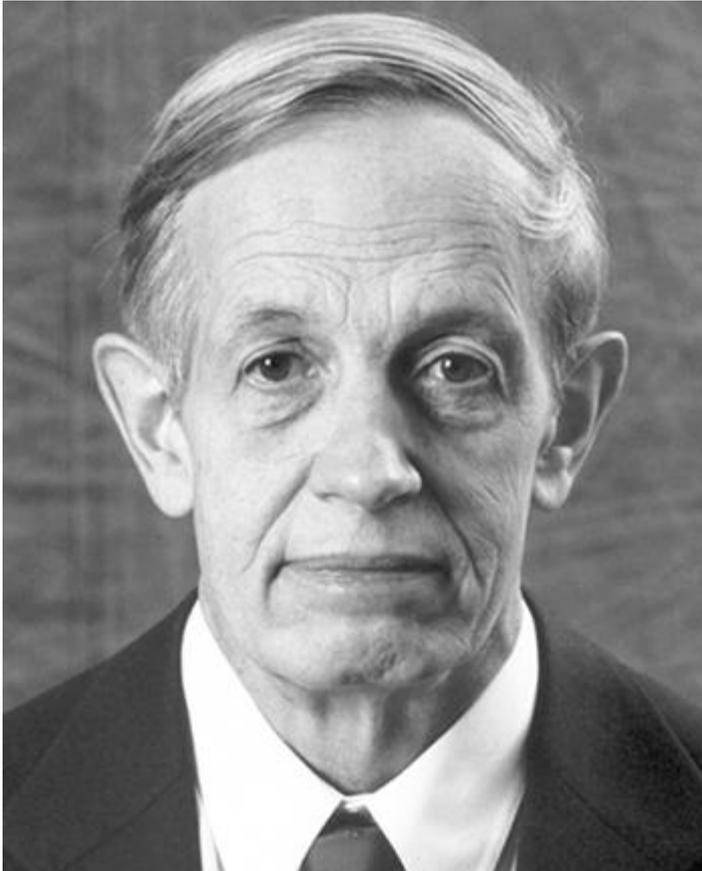
Maximum: dépend de la continuité désirée

$C^1 : 2n$ [Nash-Kuiper]

$C^k : \text{borné par } n(3n+11)/2$ [Nash, Nash-Moser]

4. Maillages anisotropes

Deux mots sur John Nash



- Théorème du plongement isométrique
- Equilibre de Nash → prix Nobel d'économie

L'**existence** est démontrée, mais ça nous dit pas **comment calculer** le plongement étant donné une surface et un champ d'anisotropie G .

4. Maillages anisotropes

L'intégration convexe et le tore plat



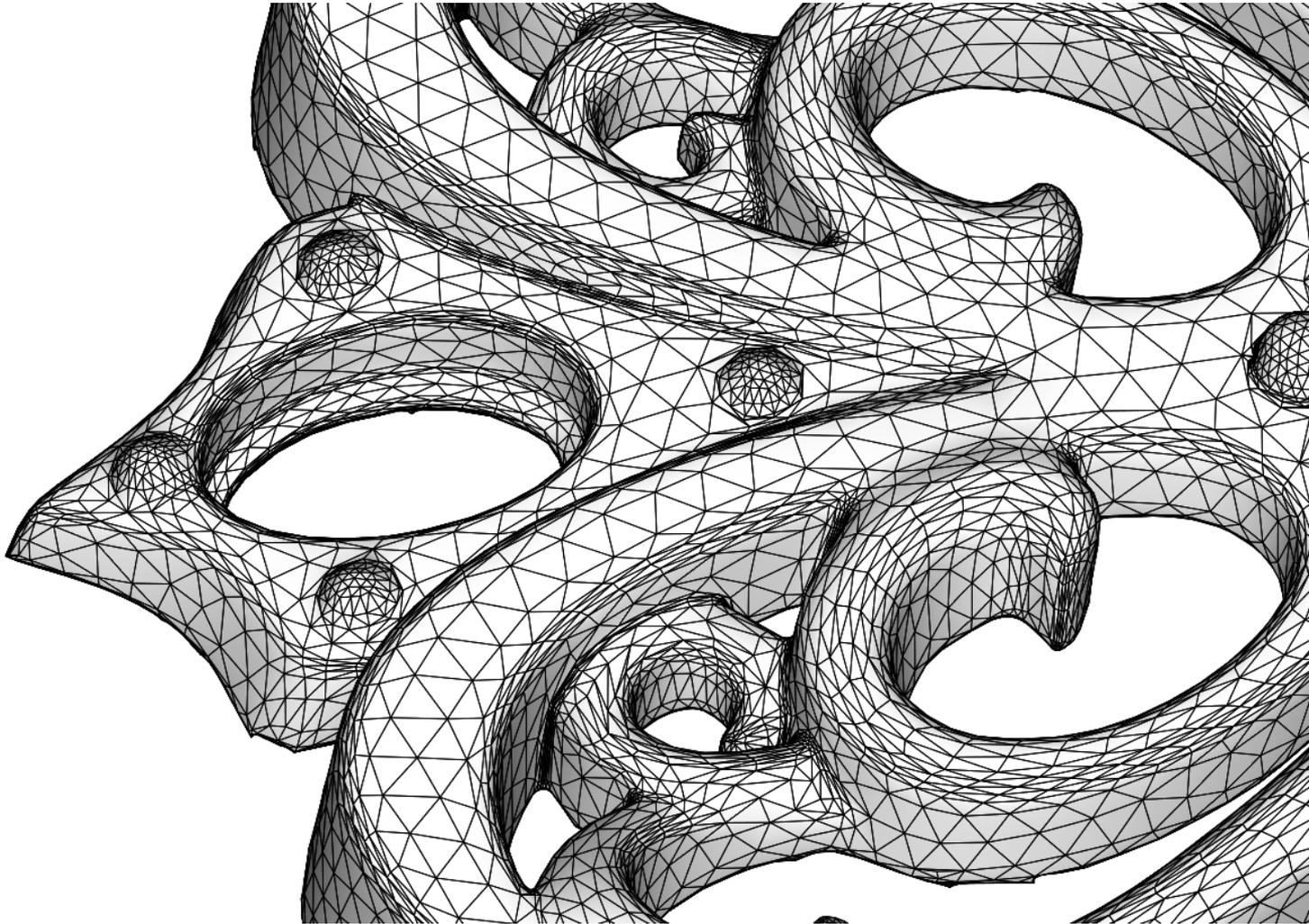
[Borelli, Jabrane, Lazarus, Thibert]

4. Maillages anisotropes – maillage scanner



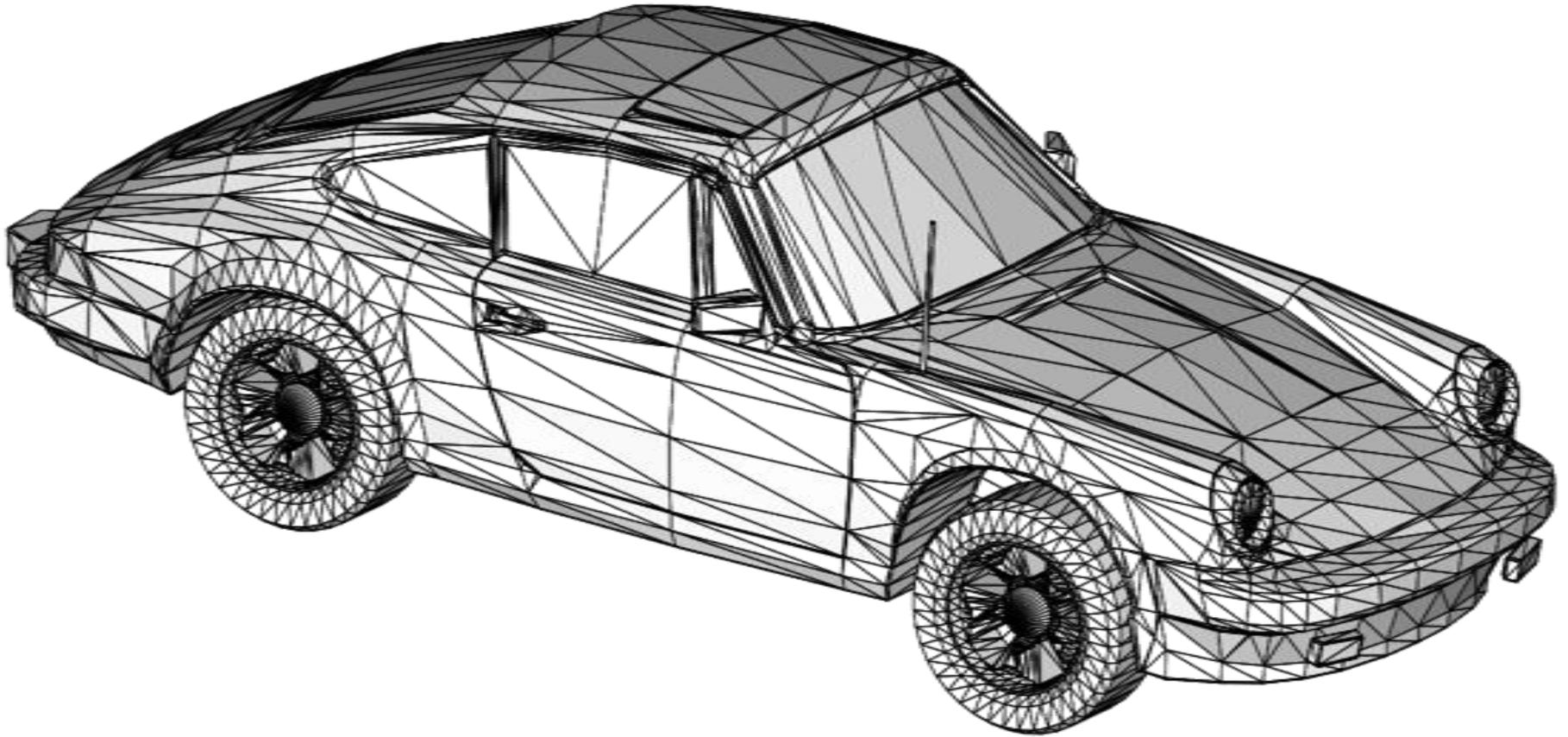
On utilise ici un plongement simple en 6d, permettant de prendre en compte la courbure

4. Maillages anisotropes – maillage scanner

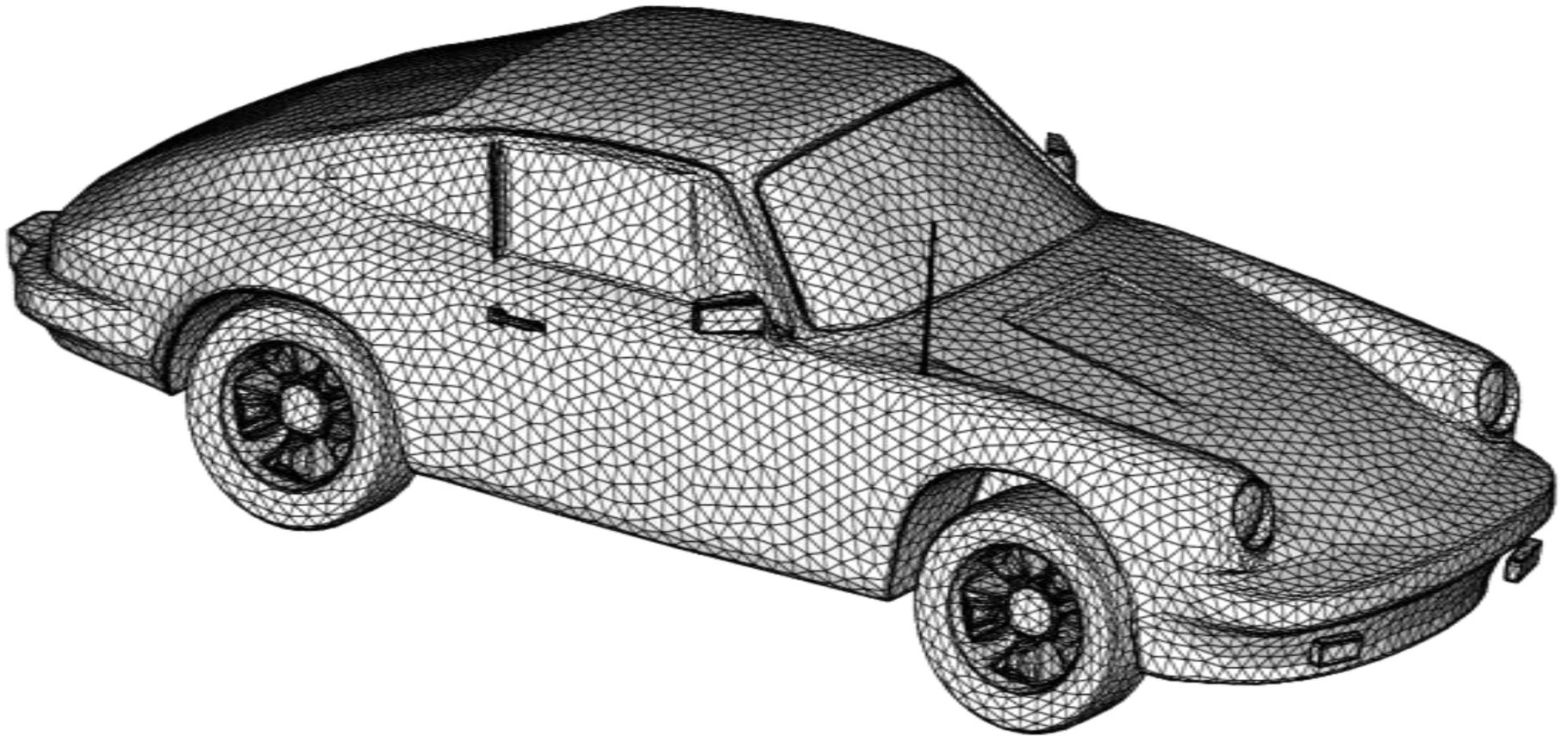


On utilise ici un plongement simple en 6d, permettant de prendre en compte la courbure

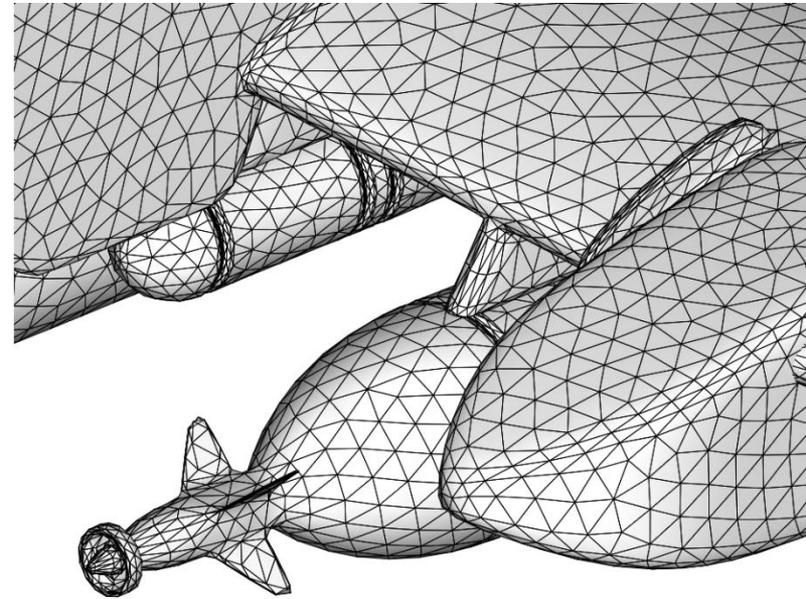
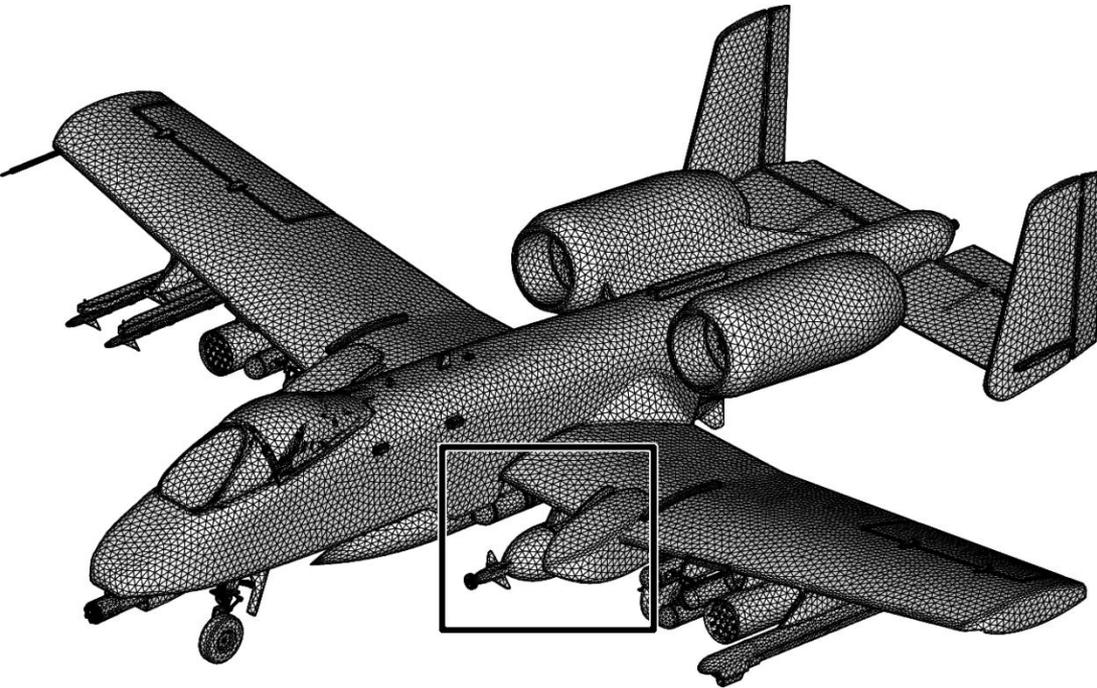
4. Maillages anisotropes – Porsche Distene



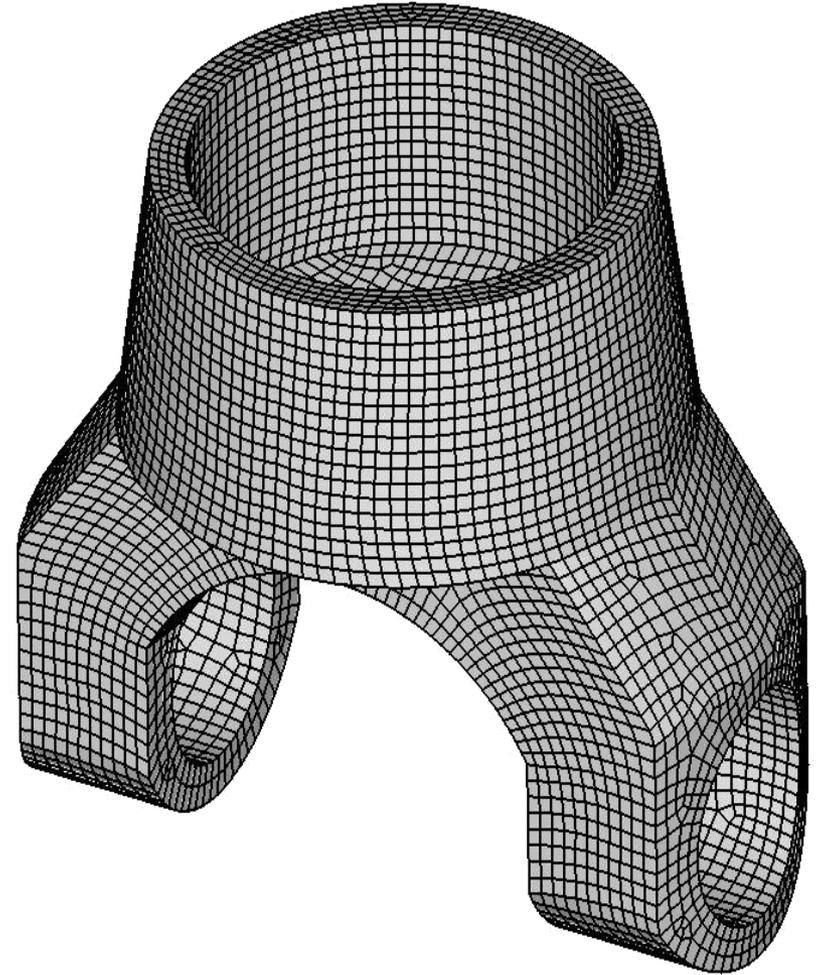
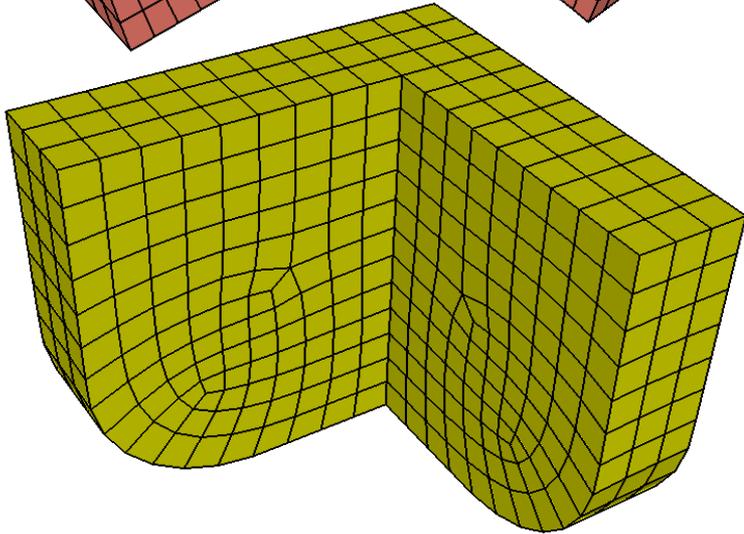
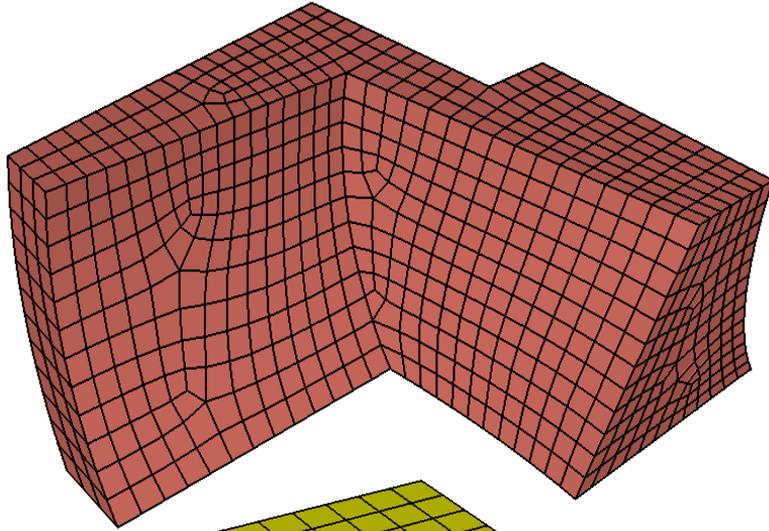
4. Maillages anisotropes – Porsche Distene



4. Maillages anisotropes – Avion Distene



5. Maillages hex-dominants



5. Maillages hex-dominants

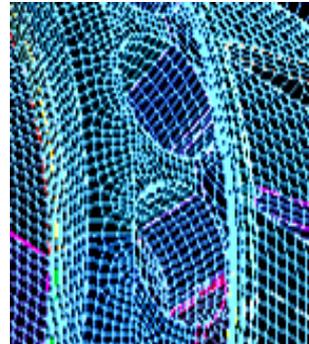
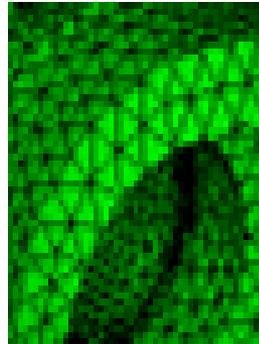
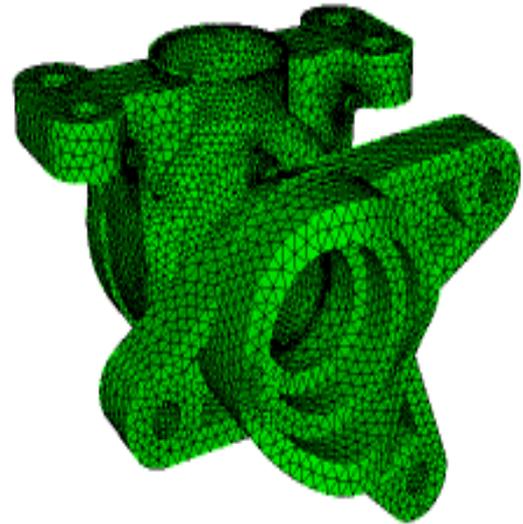
Pourquoi des cubes ?

Tet Meshing

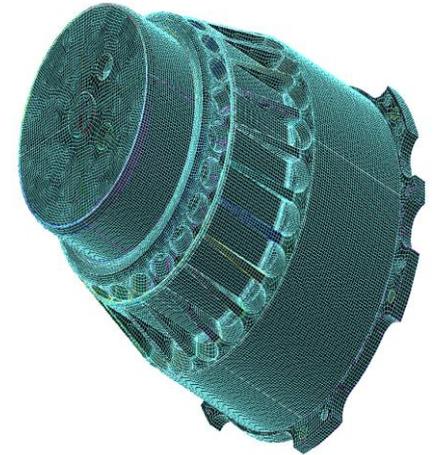
1. Fully Automated
2. Millions of elements in minutes/seconds
3. Adequate for some analysis
4. Inaccurate for other Analysis

Hex Meshing

1. Partially Automated, some Manual
2. Millions of elements in days/weeks/months
3. Preferred by some analysts for solution quality



[Matt Staten] (Sandial Labs)



5. Maillages hex-dominants

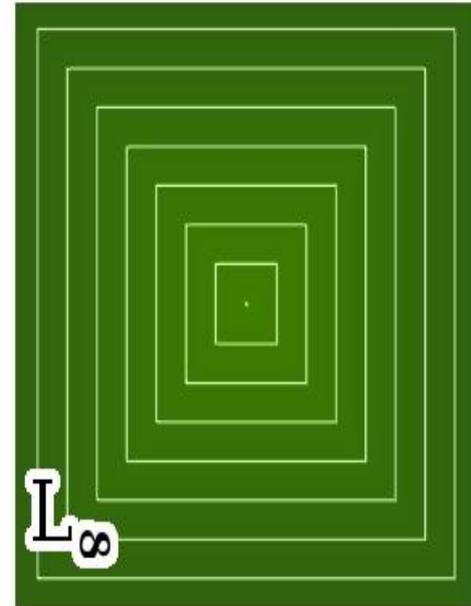
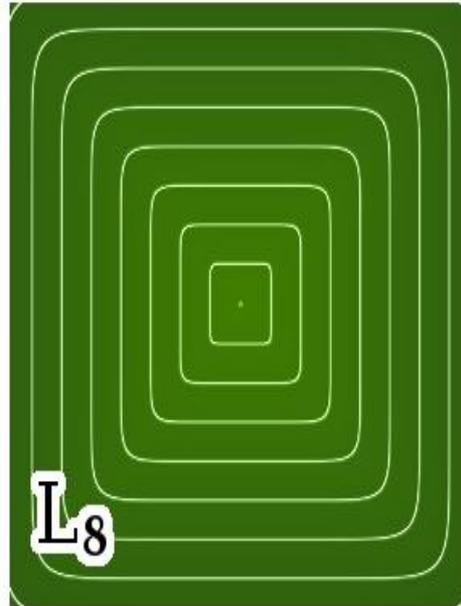
... des bulles carrées ...

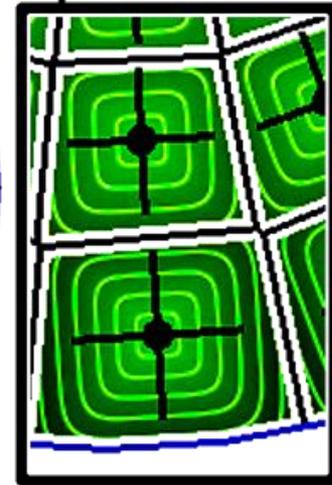
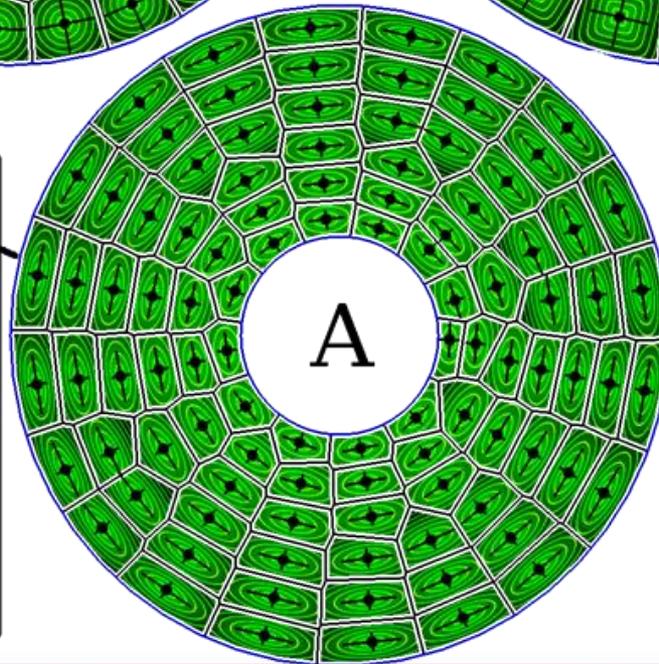
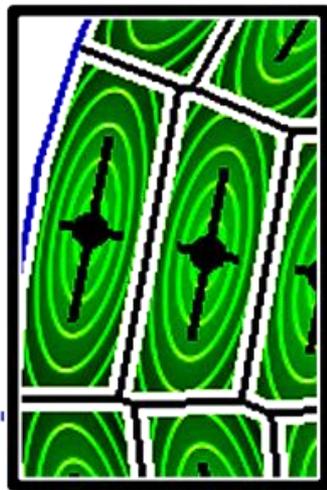
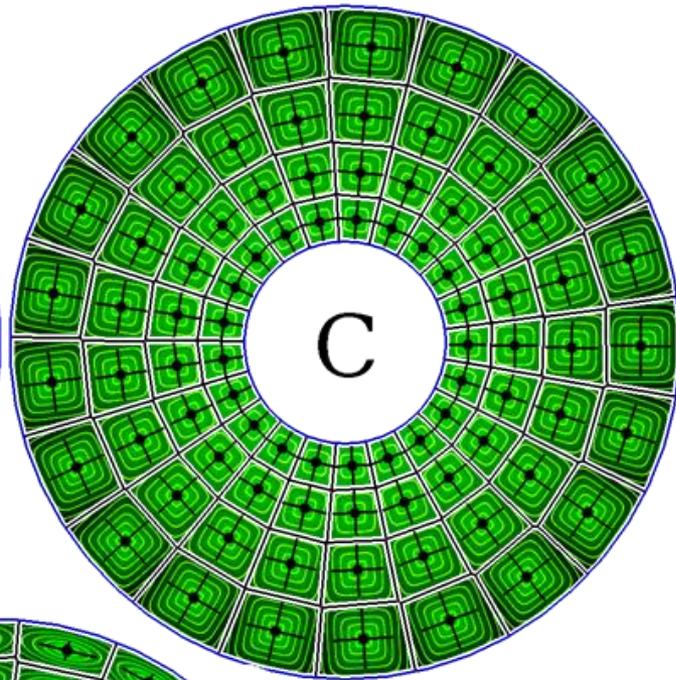
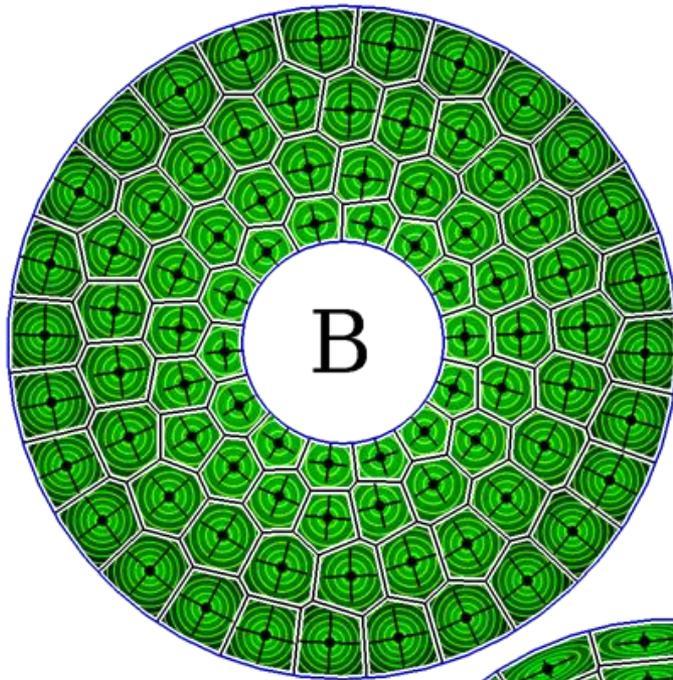
$p=2$

$p=4$

$p=8$

....





5. Maillages hex-dominants

Algorithme

CVT:

$$F = \sum_i \int_{\text{Vor}(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 dx$$

Lp CVT:
[L and Liu 2010]

$$F = \sum_i \int_{\text{Vor}(i)} \left\| \mathbf{M}(\mathbf{x}) (\mathbf{x}_i - \mathbf{x}) \right\|_p dx$$

5. Maillages hex-dominants

Algorithme

L_p CVT:

$$F = \sum_i \int_{\text{Vor}(i)} \left\| \mathbf{M}(\mathbf{x}) (\mathbf{x}_i - \mathbf{x}) \right\|_p dx$$



Anisotropie, encode l'orientation
Métrique Riemannienne $\mathbf{G} = \mathbf{M}^t \mathbf{M}$

5. Maillages hex-dominants

Algorithme

L_p CVT: $F = \sum_i \int_{\text{Vor}(i)} \|M(\mathbf{x})(\mathbf{x}_i - \mathbf{x})\|_p^p dx$

Norme L_p : $\|x\|_p = \sqrt[p]{|x|^p + |y|^p + |z|^p}$

Si p est pair: $\|x\|_p^p = x^p + y^p + z^p$

5. Maillages hex-dominants

Algorithme

$$\text{Lp CVT: } F = \sum_i \int_{\text{Vor}(i)} \left\| \mathbf{M}(\mathbf{x}) (\mathbf{x}_i - \mathbf{x}) \right\|_p dx$$

Optimisation avec L-BFGS (quasi-Newton)

For each iterate $\mathbf{X}^{(k)}$:

Compute $F(\mathbf{X}^{(k)})$ and $\nabla F(\mathbf{X}^{(k)})$

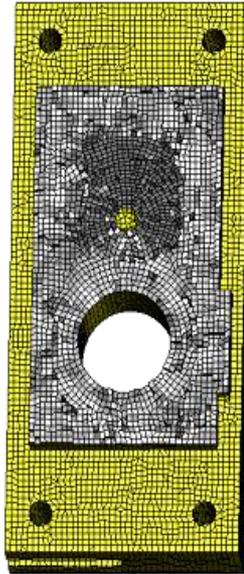
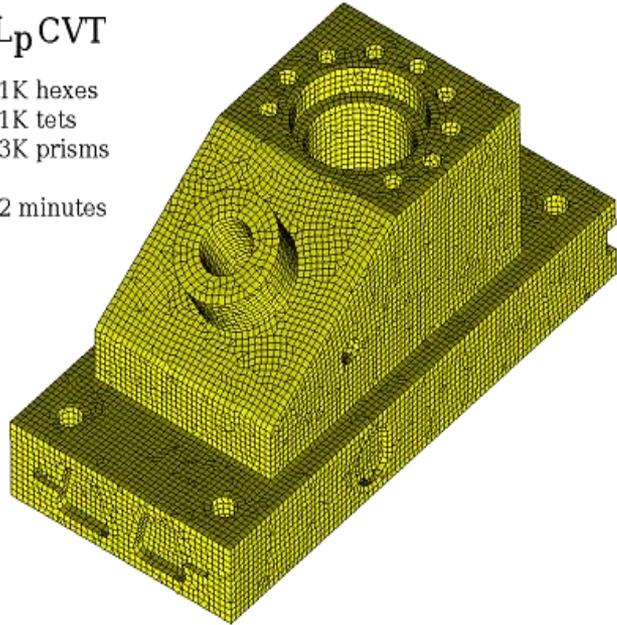
5. Maillages hex-dominants

Résultats

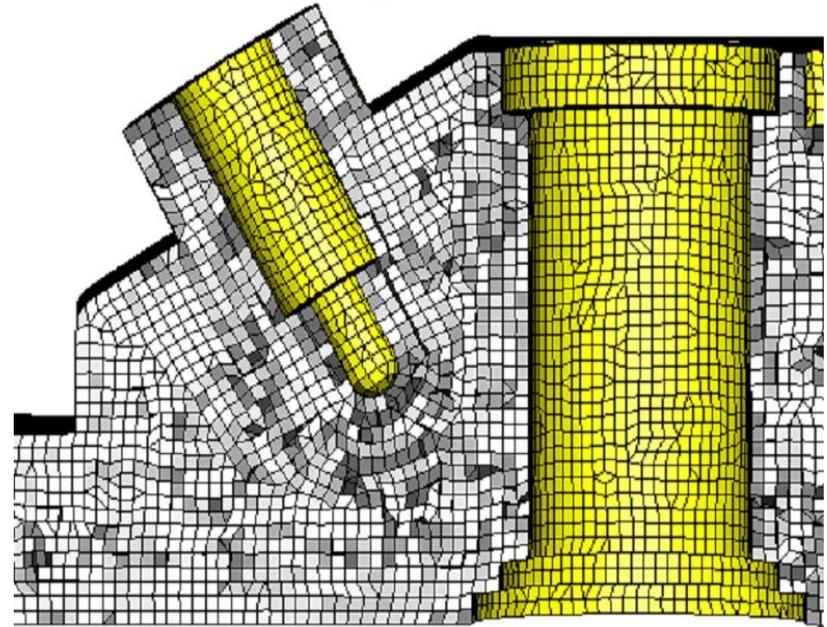
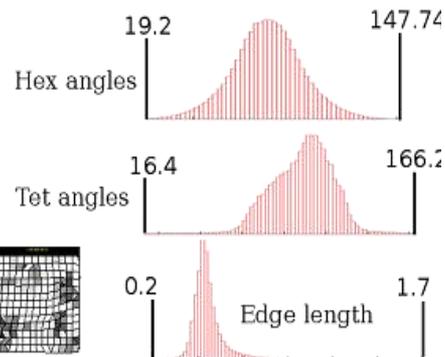
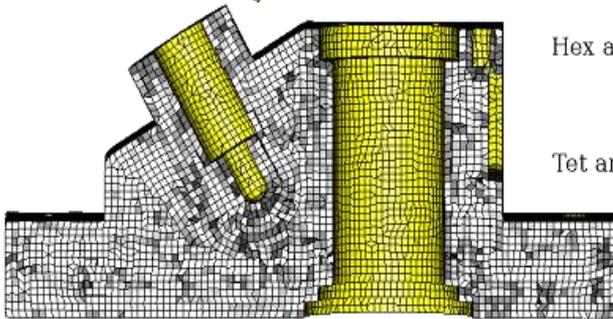
L_p CVT

81K hexes
11K tets
13K prisms

12 minutes

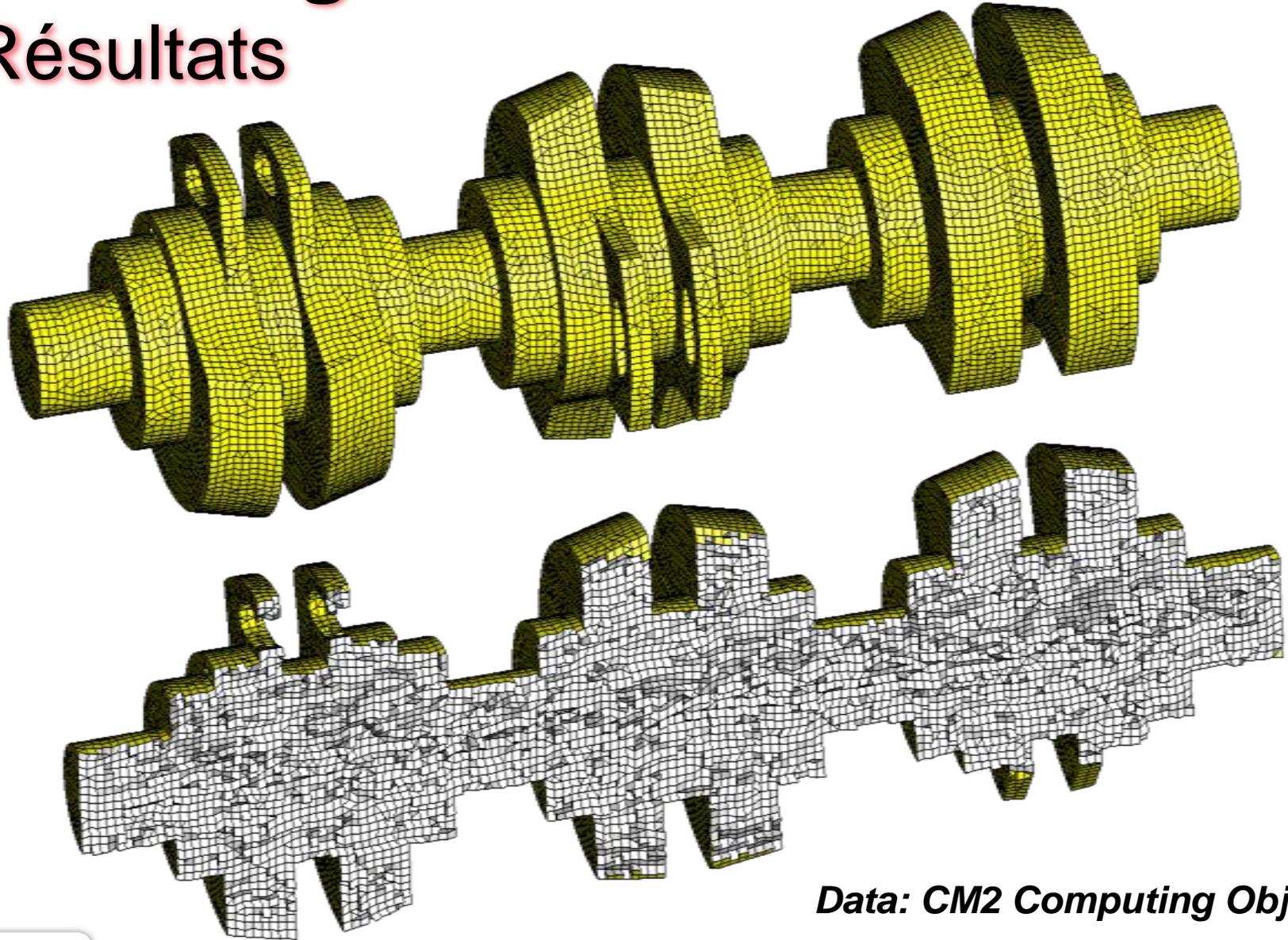


+ many other examples in paper and supplemental material.



5. Maillages hex-dominants

Résultats



Data: CM2 Computing Objects

Remerciements

Merci à Jean-Pierre Banatre pour son soutien, son enthousiasme et son aide au montage du projet GOODSHAPE.

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Tamal Dey, David Bommes, Leif Kobbelt*