Substructuring for multiscale problems

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DD 21, Rennes, June 2012

Outline

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Model Problem

Find $u \in H^1(\Omega)$, $u_{|\Gamma_D} = 0$ $\qquad \Omega \subset \mathbb{R}^d \quad d = 2, 3$

$$\int_\Omega \boldsymbol{\alpha} \nabla u \cdot \nabla v \, dx \;=\; f(v) \qquad \forall v \in H^1(\Omega),\ v_{|\Gamma_D} = 0$$

strongly varying coefficient

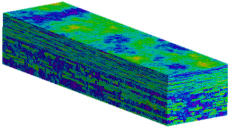
$\boldsymbol{\alpha} \in L^\infty_+(\Omega)$ (uniformly positive)

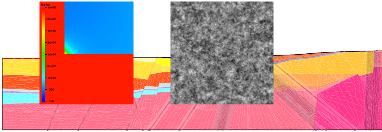
Conditioning of FE system:

$$\sim \underbrace{\sup_{x,y\in\Omega} \left(\frac{\alpha(x)}{\alpha(y)}\right)}_{\mathcal{O}(10^3) - \mathcal{O}(10^{10})} h^{-2}$$

Goal:
iterative solvers, robust in h and $\boldsymbol{\alpha}$

towards applications:
porous media flows
oil reservoir simulation
elasticitiy with
heterogeneous material
nonlinear magnetostatics

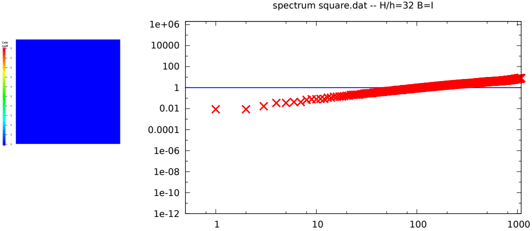


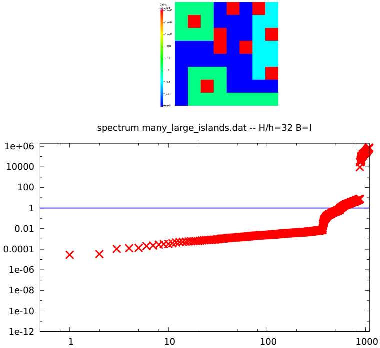


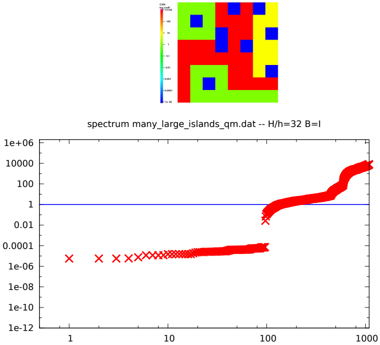
Spectral properties – I

unit square
uniform mesh $h = 1/32$
pure Neumann problem

$\sigma(K_{h,\alpha})$



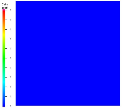


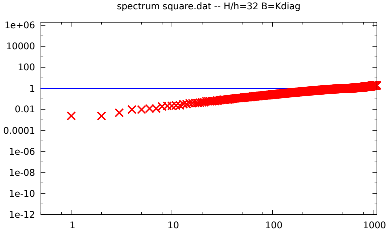


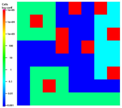
Spectral properties – II

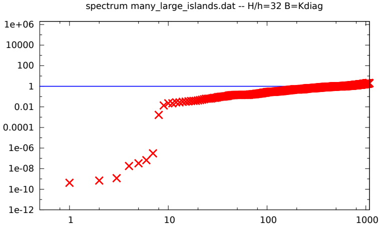
unit square
uniform mesh $h = 1/32$
pure Neumann problem

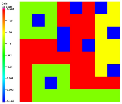
$\sigma(\mathrm{diag}(K_{h,\alpha})^{-1}\, K_{h,\alpha})$

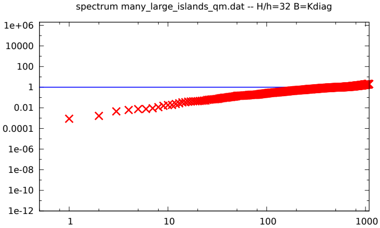












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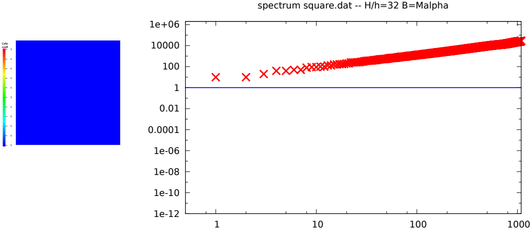
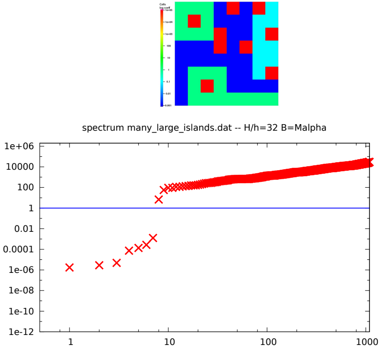
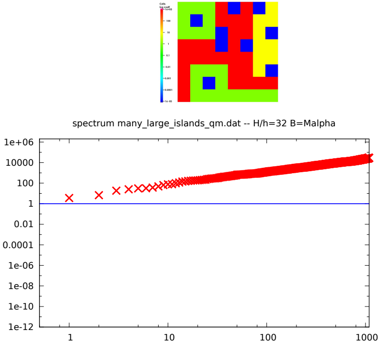
$$\sigma(M_{h,\alpha}^{-1} K_{h,\alpha})$$







Weighted Poincaré Inequality (WPI)

domain D, coefficient $\alpha \in L^\infty_+(D)$

Definition

$C_{P,\alpha}(D)$ smallest constant such that $\forall u \in H^1(D)$:

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^2(D),\alpha} \leq \underbrace{C_{P,\alpha}(D)\,\mathrm{diam}(D)}_{\approx \lambda_2(M_{h,\alpha}^{-1} K_{h,\alpha})^{-1/2}} |u|_{H^1(D),\alpha}$$

where [Chua] [Chua & Wheeden] [Veeser & Verfürth] [Zikhov] ...

$$\|v\|^2_{L^2(D),\alpha} = \int_D \alpha |v|^2\, dx, \qquad |v|^2_{H^1(D),\alpha} = \int_D \alpha |\nabla v|^2\, dx$$

infimum attained at **weighted average**: $c = \overline{u}^{D,\alpha} = \frac{\int_D \alpha\, u\, dx}{\int_D \alpha\, dx}$

For **quasi-monotone** coefficients (details soon):

$$C_{P,\alpha}(D) \leq C \quad \textbf{independently of contrast}$$

[P. & Scheichl '11, '12*] early results by [Galvis & Efendiev '10]

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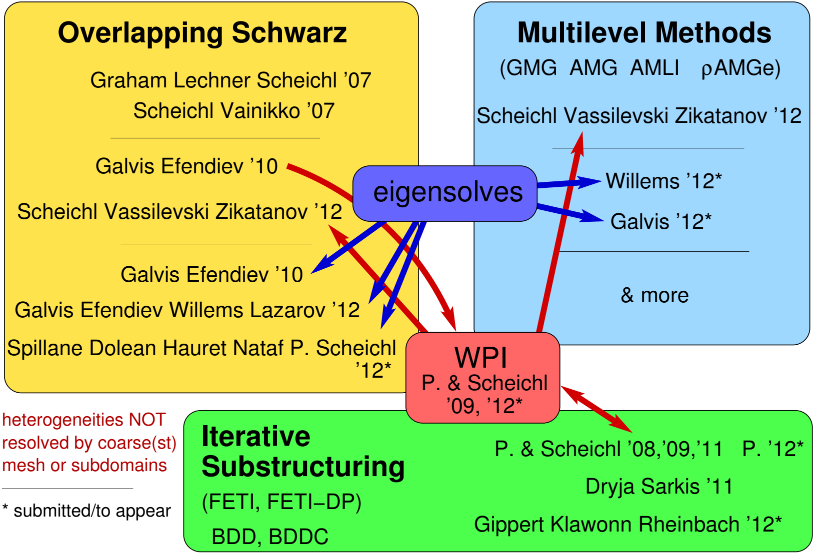
Preconditioners for multiscale problems



Minisymposium M13 (Tu 16.30, We 10.30) links to upscaling

Outline

Quasi-monotone coefficients

Let $\alpha \in L^\infty_+(D)$ be **piecewise constant** w.r.t. partition

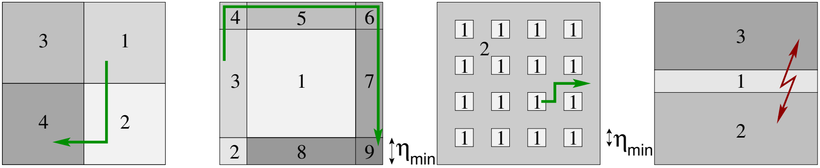
$\{Y_k\}_{k=1}^n$ (Y_k connected Lipschitz) $\alpha_k := \alpha_{|Y_k} = \text{const}$

$Y_L \ldots$ subregion with **largest** coefficient

Definition

α is **quasi-monotone** on D iff
for each k there exists a **path** P_{kL} from Y_k to Y_L:

- subsequent subregions share common $(d-1)$-dim. **facet**
- coefficient **does not decrease** along the path

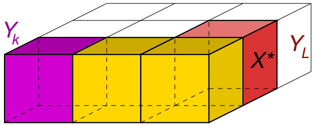


(a) (b) (c) (d)

α **quasi-monotone**

P_{kL} corresp. path from Y_k to Y_L

$X^* \subset \overline{Y}_L$



Definition

c_{kL} best constant:

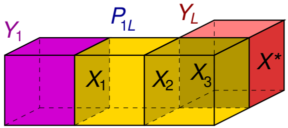
$$\|u - \overline{u}^{X^*}\|^2_{L^2(Y_k)} \leq c_{kL}\,\mathrm{diam}(D)^2\,|u|^2_{H^1(P_{kL})} \qquad \forall u \in H^1(P_{kL})$$

Lemma

$$\|u - \overline{u}^{X^*}\|^2_{L^2(D),\alpha} \leq \Big(\sum_{k=1}^{n} c_{kL}\Big)\,\mathrm{diam}(D)^2\,|u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D)$$

$$\Longrightarrow \quad C_{P,\alpha}(D)^2 \leq \Big(\sum_{k=1}^{n} c_{kL}\Big)$$ *independent of constrast!*

Explicit dependence on geometric scales



Lemma

P_{kL} ... *path from* Y_k *to* Y_L $\qquad$ X_i ... *interfaces or* X^*

$$c_{kL} \leq 2 \sum_{Y_\ell \subset P_{kL}} \sum_{X_i \subset \partial Y_\ell} \frac{|Y_k|}{|Y_\ell|} \frac{\mathrm{diam}(Y_\ell)^2}{\mathrm{diam}(D)^2} C_P(Y_\ell, X_i)$$

Lemma

α *quasi-monotone on* D

$\{Y_\ell\}$ *form shape regular partition, not extremely long paths*

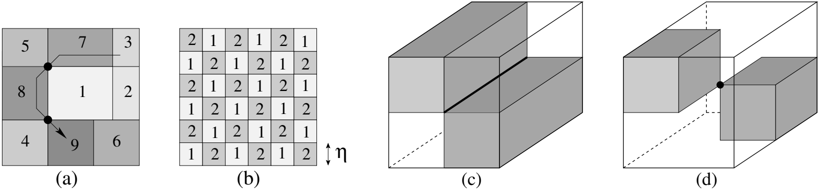
$$C_{P,\alpha}(D)^2 \lesssim s_{\max} \frac{\mathrm{meas}(D)}{\mathrm{diam}(D)^2\, \eta_{\min}^{d-2}} \lesssim \left(\frac{\mathrm{diam}(D)}{\eta_{\min}}\right)^{d-1}$$

WPI for FE functions

Definition

α **type-m quasi-monotone** on D iff
for each k there exists a **path** P_{kL} from Y_k to Y_L:

- subsequent subregions share common m-dim. **facet**
- weights within the path **do not decrease**



Analogous statements hold true on FE space $V^h(D)$,
but constants depend on h:

$m = d - 2$: $1 + \log(\eta/h)$

$m = d - 3$: η/h

More details:
[P. & Scheichl: IMAJNA 2012 (to appear)]

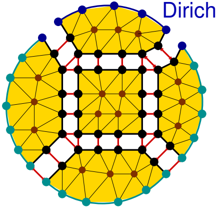
Outline

Finite **E**lement **T**earing and **I**nterconnecting

FETI: [Farhat & Roux]

FETI-DP: [Farhat, Lesoinne, Le Tallec, Pierson, Rixen]

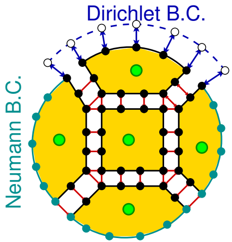
TFETI: [Dostál, Horák, Kučera], [Of, Steinbach]



$$\begin{bmatrix} K_1 & & 0 & B_1^\top \\ & \ddots & & \vdots \\ 0 & & K_N & B_N^\top \\ B_1 & \cdots & B_N & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_N \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \\ 0 \end{bmatrix}$$

compactly: $$\begin{bmatrix} K & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

TFETI



coarse DOF: p.w. constants
("rigid body modes")

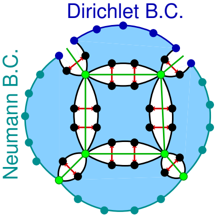
solve $P^\top B K^\dagger B^\top \widetilde{\lambda} = \widetilde{d}$

precond. $P B_D S B_D^\top$

$$P = I - Q G \underbrace{(G^\top Q G)^{-1}}_{\text{coarse solve}} G^\top$$

B_D, Q contain scalings

FETI-DP



primal DOF: vertex values
(weighted) edge averages
???

solve $B \widetilde{K}^{-1} B^\top \lambda = d$

precond. $B_D S B_D^\top$

$\widetilde{K}^{-1}$: block Cholesky
coarse solve + subdomain solves

B_D contains scalings

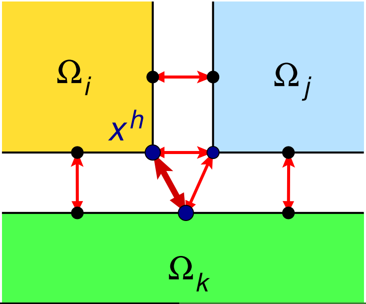
Definition of jump operators

jump operators:

$$B : W \to U \qquad W \ldots \text{torn FE space}$$
$$B_D : W \to U \qquad U \ldots \text{Lagrange multiplier space}$$

x^h ... interface DOF (node) on $\partial\Omega_i \cap \partial\Omega_k$:

$$(B\,w)_{ik}(x^h) \;=\; w_i(x^h) - w_k(x^h)$$



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x^h ... interface DOF (node) on $\partial\Omega_i \cap \partial\Omega_k$:

$$(B_D w)_{ik}(x^h) \;=\; \frac{1}{\sum\limits_{j \in \mathcal{N}_{x^h}} \rho_j(x^h)} \Big[\rho_k(x^h)\, w_i(x^h) - \rho_i(x^h)\, w_k(x^h) \Big]$$

scalings $\rho_j(x^h)$ possible choices discussed later on

- per subdomain j
- per interface DOF x^h

partition of unity property
averaging property: $I - B_D^\top B \;=\; E_D : W \to \widehat{W}$

Analysis framework

For FETI, FETI-DP (also BDD, BDDC):

- $\lambda_{\min} \geq 1$
- $\lambda_{\max} \leq \sup_{w \in W_{\text{sub}}} \frac{|\overbrace{B_D^\top B}^{P_D} w|_S^2}{|w|_S^2}$ $\qquad W_{\text{sub}} \subset W$ suitable

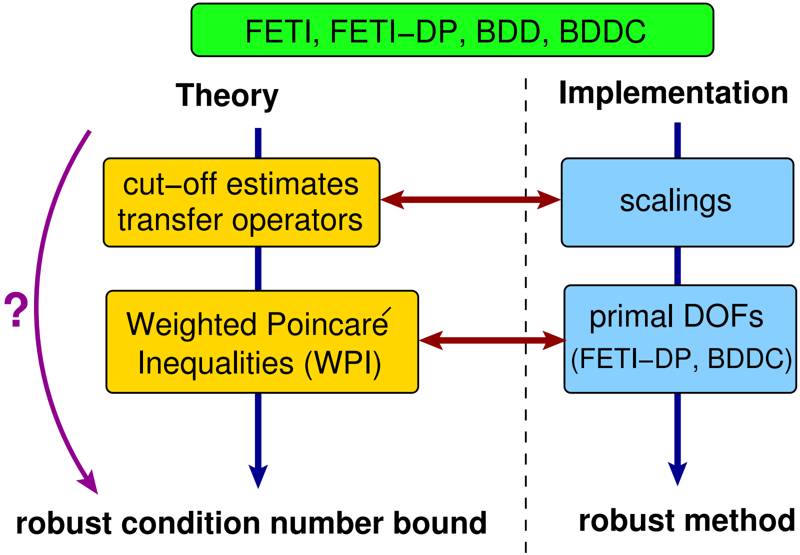
$|\cdot|_S$ ... Schur complement norm on W

Common technique:

- **splitting** into **subdomain face, edge, vertex** contributions using **cut-off** functions
- **transfer operators** (Sobolev extension + Scott-Zhang)
- **Poincaré inequality**

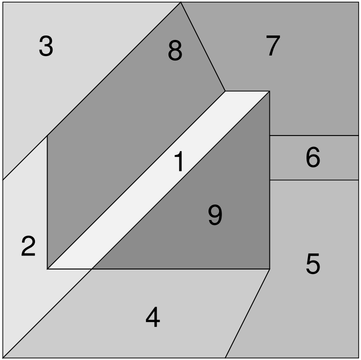
[Mandel & Tezaur] [Klawonn & Widlund] [Klawonn, Widlund, Dryja]
[Dohrmann, Mandel, Tezaur] [Mandel & Sousedik] [Klawonn, Rheinbach, Widlund]

Theory vs. Implementation



Outline

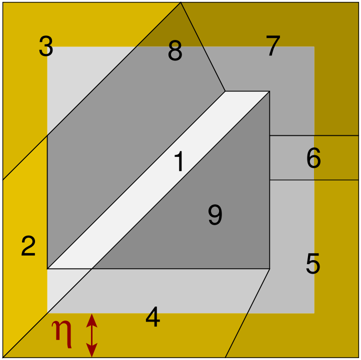
subdomain Ω_i



Assumptions througout:

- patch decomposition $\rightsquigarrow$ globally conforming mesh $\mathcal{T}^\eta$
- patch decomposition $\mathcal{T}^\eta(\Omega_{i,\eta})$ quasi-uniform
- variation on single patch $\leq c_{\text{noise}}$ small

boundary layer $\Omega_{i,\eta}$



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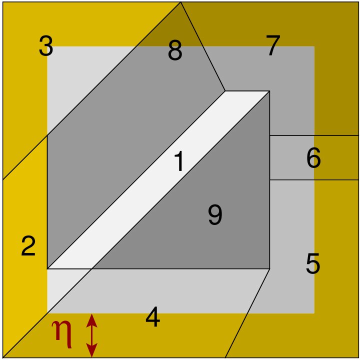
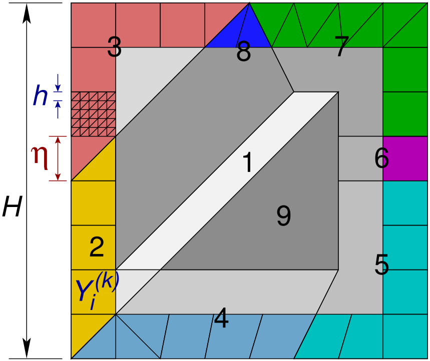
boundary layer $\Omega_{i,\eta}$ **patch decomposition:** $\overline{\Omega}_{i,\eta} = \bigcup_k \overline{Y}_i^{(k)}$





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Need good scalings

$\rho_j(x^h)$	**Theory**	**Implementation**
(a)	1	1 (multiplicity scaling)
(b)	$\alpha_{\Omega_j}^{\max}$	$\max(K_j^{\text{diag}})$
(c)	$\max\limits_{\tau\subset\Omega_j : x^h\in\overline{\tau}} \alpha_{\|\tau}$	$K_j^{\text{diag}}(x^h)$ (stiffness scaling)
(d)	$\max\limits_{Y_j^{(k)} : x^h\in\overline{Y}_j^{(k)}} \alpha_{Y_j^{(k)}}^{\max}$	processed stiffness scaling ?
(e)	new promising technique, C. Dohrmann's talk, Mo, M13	

Choice (a): does not work for certain jumps across interfaces

Choice (b): κ may depend on $\max\limits_i \dfrac{\alpha_{\Omega_i}^{\max}}{\alpha_{\Omega_i}^{\min}}$

Choice (c): κ may depend on oscillations of $\boldsymbol{\alpha}$
or deteriorate for ragged interfaces!

Intermediate result

Lemma (P. & Scheichl '11, P. '12*)

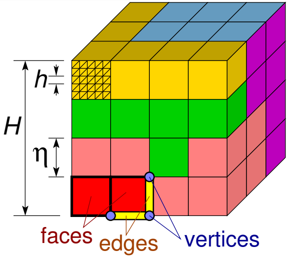
For (theoretical) choice (d), $\rho_j(x^h) = \max_{Y_j^{(k)}: x^h \in \overline{Y}_j^{(k)}} \alpha^{\max}_{Y_j^{(k)}}$

$\forall w \in W$:

$$|P_D w|_S^2 \;\le\; C\, c_{noise} \sum_{j=1}^{N} \Big[(1 + \log(\tfrac{\eta}{h}))^2\, |w_j|^2_{H^1(\Omega_{j,\eta}),\boldsymbol{\alpha}} + (1 + \log(\tfrac{\eta}{h}))\, \eta^{-2}\, \|w_j\|^2_{L^2(\Omega_{j,\eta}),\boldsymbol{\alpha}} \Big]$$

Proof:

- **finer** splitting into patch globs
- transfer operators (carefully!)
- conventional cut-off estimates



(d) difficult to mimic in practice
(edge detection, TV minimization)

WPI in boundary layers

Assumption:

- "nice" subdomains Ω_j
- paths not extremely long

Then:

- $C_{P,\alpha}(\Omega_j)^2 \lesssim \left(\frac{H}{\eta}\right)^{d-1}$
- $C_{P,\alpha}(\Omega_{j,\eta})^2 \lesssim \left(\frac{H}{\eta}\right)^{d-2}$

Outline

Condition number bounds – I

TFETI, $Q = B_D \, S \, B_D^\top$ [P. & Scheichl '08, '11, P. '12]

Theorem

α ***constant*** *in boundary layers* $\Omega_{i,\eta}$

$$\kappa \;\leq\; C \, c_{noise} \left(\frac{H}{\eta}\right)^2 \left(1 + \log\left(\frac{\eta}{h}\right)\right)^2$$

improves *to* H/η *if* α *inside* ***larger*** *than in* $\Omega_{i,\eta}$

Theorem

α ***quasi-monotone*** *in boundary layers* $\Omega_{i,\eta}$

$$\kappa \;\leq\; C \, c_{noise} \left(\frac{H}{\eta}\right)^d \left(1 + \log\left(\frac{\eta}{h}\right)\right)^2$$

Similar: $Q = Q_{\mathrm{diag}}$; type-m quasi-monotone

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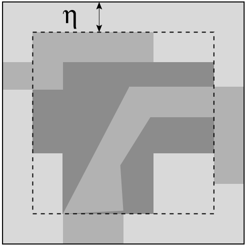
Similar: $Q = Q_{\mathrm{diag}}$; type-m quasi-monotone

artificial coefficient α^{art}:

$$\alpha^{\text{art}} = \alpha \quad \text{in } \Omega_{j,\eta}$$
$$\alpha^{\text{art}} \leq \alpha \quad \text{elsewhere}$$



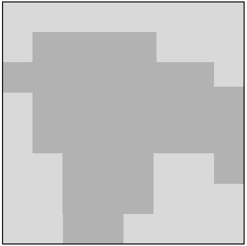
Enough to have

$$\inf_{c \in \mathbb{R}} \|w - c\|^2_{L^2(\Omega_{j,\eta}),\alpha} \leq C_j^* H^2 |w|^2_{H^1(\Omega_j),\alpha^{\text{art}}}$$

Theorem

α^{art} quasi-monotone on Ω_j:

$$\kappa \leq C\, c_{noise} \left(\frac{H}{\eta}\right)^{d+1} \left(1 + \log\left(\frac{\eta}{h}\right)\right)^2$$



[P. & Scheichl '08,'11] [Dryja & Sarkis '11] [Gippert, Klawonn, Rheinbach '12*]

Necessity of quasi-monotonicity?

Q: is quasi-monotonicity is necessary for the robust of TFETI ?

Conjecture

TFETI is robust iff for each subdomain, there exists a quasi-monotone artificial coefficient.

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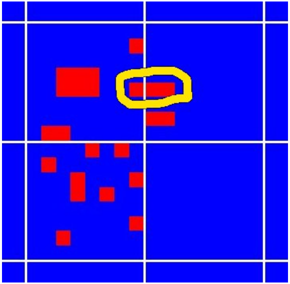
Conjecture

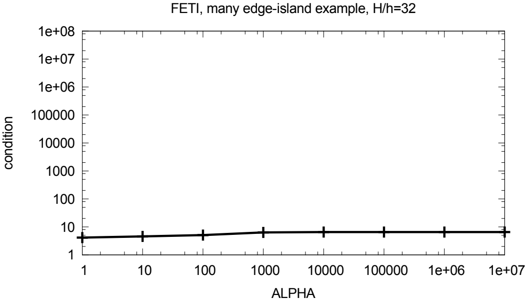
TFETI is robust iff for each subdomain, there exists a quasi-monotone artificial coefficient.

That's wrong!

TFETI has more robustness properties!

Docking inclusions can be elminated!



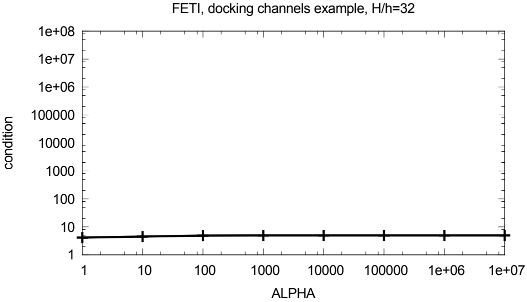


Theoretical reason: [work in progress]

We actually have to estimate

$$\int_{\Omega_{i,\eta}} \min(\alpha_i,\, \alpha_{\text{neighbors}})\, |\ldots|^2 \, dx$$

Even docking channels can be elminated!



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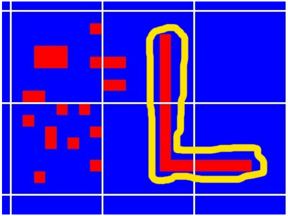
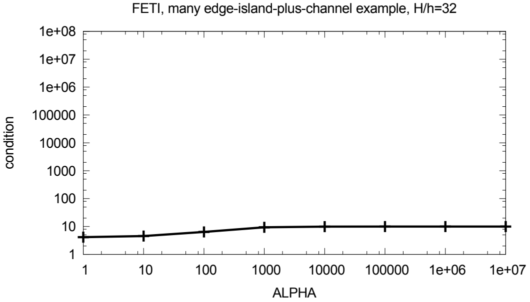
Face inclusions can be elminated!

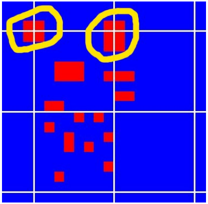


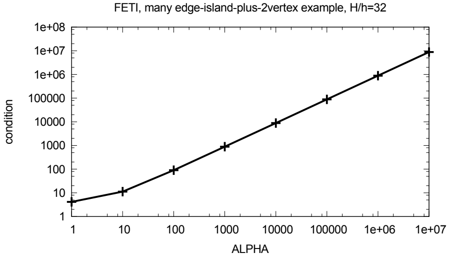


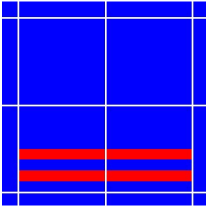
Theoretical reasons: [work in progress]

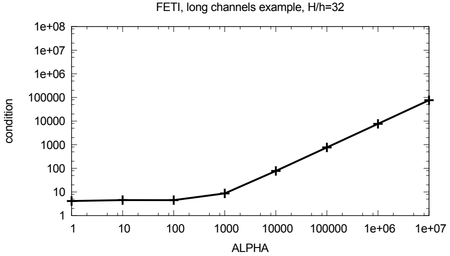
- Special cut-off function [Graham, Lechner, Scheichl '07]
- Special transfer operator, L^2_α , H^1_α-stable for binary media

E.g. **two vertex inclusions** or **long channels**









Outline

Choice of primal DOFs

Choice of **primal DOFs** should

- ensure that $\widetilde{K}$ is **invertible**
- make **WPI** applicable (on neighborhoods of edges/faces)

With tools from above this will allow for robustness analysis

---

Adapted tool:
WPI with weighted averages [P., Sarkis, Scheichl, DD20 proc.]

$$\|u - \overline{u}^{X,\widehat{\alpha}}\|_{L^2(D),\alpha} \leq C_{P,\alpha}(D, X, \widehat{\alpha})\,\mathrm{diam}(D)\,|u|_{H^1(D),\alpha}$$

Essential requirements:

- α quasi-monotone
- averaging manifold X must see largest coefficient
- $\widehat{\alpha}$ and α have to match

Geometry dependence can again be made explicit

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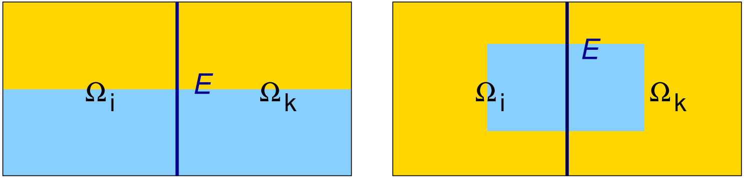
Weighted edge averages?

Simple examples:

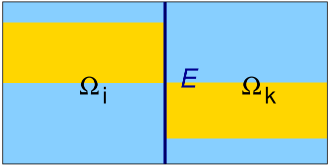


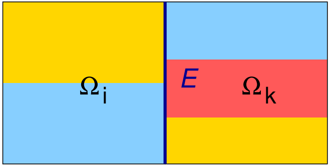
Primal DOF associated to edge E:

$$\overline{u}^{E,\widehat{\alpha}} := \frac{\int_E \widehat{\alpha}\, u\, ds}{\int_E \widehat{\alpha}\, ds} \quad \text{or} \quad \overline{u}^{E,\widehat{\alpha},\mathrm{alg}} := \frac{\sum_{x^h \in \overline{E}} \widehat{\alpha}(x^h)\, u(x^h)}{\sum_{x^h \in \overline{E}} \widehat{\alpha}(x^h)}$$

[Klawonn & Rheinbach, 2006]

What to do here?





Problems:

- Primal DOF should have **same** weight for both sides
- Usage of multiple primal DOFs $\rightsquigarrow$ dead end?

Recall: for edge E, we actually have to bound

$$\int_{\Omega_{i,\eta} \cap U_E} \text{"}\min(\alpha_i, \alpha_k)\text{"}\, |\ldots|^2\, dx$$

If **minimum coefficient** has quasi-monotone extensions then

$$\widehat{\alpha}(x^h) = \min(K_i^{\text{diag}}(x^h), K_k^{\text{diag}}(x^h))$$

(algebraic average) does the job, at least robust w.r.t. contrast!

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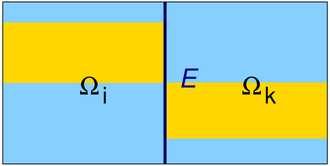

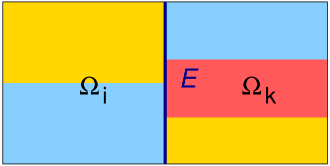

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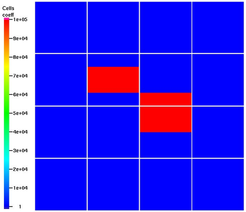
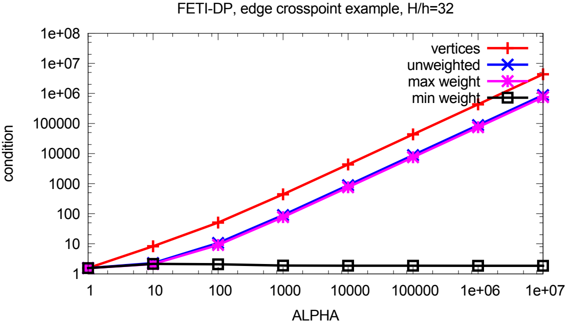
Numerical test





In this talk:

- Weighted Poincaré Inequalities
- Robustness theory for TFETI
- Robustness theory for FETI-DP (yet to be completed)

Theory **and** testing guides to more insight / new methods!

Open problems:

- Choice of scalings
- Choice of primal constraints
- Incorporation of eigensolves

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monograph:
Finite and Boundary Element Tearing and Interconnecting Solvers for Multiscale Problems
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Interconnecting...

THANKS FOR YOUR ATTENTION

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